Journal of Applied Logic 11 (2013) 373–385

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

The sure thing principle, dilations, and objective probabilities ☆

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ARTICLE INFO

Article history:
Available online 13 August 2013

Keywords:
Sure Thing Principle
Axiom P2
Imprecise probabilities
Dilation
Partial ordering
Objective probabilities
Savage's system
Partial acts

ABSTRACT

The common theme that unites the four sections is STP, the *sure thing principle*. But the paper can be divided neatly into two parts. The first, consisting of the first two sections, contains an analysis of STP as it figures in Savage's system and proposals of changes to that system. Also possibilities for partially ordered acts are considered. The second, consisting of the last two sections, is about imprecise probabilities, dilations and objective probabilities. Variants of STP are considered but this part is self-contained and can be read separately. The main claim there is that dilations, which can have extremely counterintuitive consequences, can be eliminated by a more careful analysis of the phenomenon. It outlines a proposal of how to do it. Here the concept of objective probabilities plays a crucial role.

1. The sure thing principle, Savage's system, and partial acts

The sure thing principle, henceforth abbreviated STP, was introduced by Savage in his seminal work [10] by means of the following story:

> A businessman contemplates buying a. certain piece of property. He considers the outcome of the next presidential election relevant to the attractiveness of the purchase. So, to clarify the matter for himself, he asks whether he would buy if he knew that the Republican candidate were going to win, and decides that he would do so. Similarly, he considers whether he would buy if he knew that the Democratic candidate was going to win, and again finds, that he would do so. Seeing that he would buy in either event, be decides that he should buy, even though he does not know which event obtains, or will obtain, as we would ordinarily say. It is all too seldom that a decision can be arrived at on the basis of the principle used by this businessman, but except, possibly for the assumption of simple ordering, I know of no other extralogical principle governing decisions that finds such ready acceptance.
>
> [10, p. 21][1]

☆ Thanks are due to many people: to Jeff Helzner, for organizing the conference, editing the volume, and patiently waiting for my final draft; to the participants – for making this an enjoyable high quality conference; to Teddy Seidenfeld – for his observations, in long email exchanges, which forced me to think harder on the subject (he also pointed out to me additional works that are relevant to the points I am discussing, but which I could not include in the present paper and may have to address in future works on these topics); to Anubav Vasudevan, Sidney Felder, Liu Yang, Rush Stewart and all the participants of the Columbia Formal Epistemology workshop, for the occasion to present some of these ideas and to get their useful reactions; and to Isaac Levi for fruitful exchanges. I would also like to mention Horatio Arlo-Costa, no longer with us; his instructive observation on related material were quite helpful at an earlier stage of this work.

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[1] All page numbers for Savage's book refer to the 1972 revised edition.

1570-8683/$ – see front matter © 2013 Published by Elsevier B.V.
http://dx.doi.org/10.1016/j.jal.2013.06.003

Underlying the scenario is the obvious assumption that the acts in question – buying the property or not buying it – have no effect whatsoever on the result of the election. They do not change the likeliness of any possible outcome.

Let us recall that Savage's framework is based on: (i) A set S of *states of the world* and a Boolean algebra of *events* (or propositions) that are sets of states, (ii) A set of *acts*, where each act is a function, f, that associates with each state, s, an *outcome*, $f(s)$ – the result of the agent's performing the act in a world in state s, and (iii) The agent's preference relation, which is a simple ordering (i.e., total preorder) of the acts.[2,3] There is a crucial assumption that the obtaining of any state is an objective fact, which is unaffected by the agent's act. We can make it even more obvious, by modifying a little the story above: Assume that the election took place already, and the businessman does not know the result – say he is on a trip in a remote region, but he has the option of sending a message to his office instructing his staff to carry out the transaction. Given STP, he can do so without waiting to find which candidate won.

The above scenario concerns strict preference for buying the property, but the same logic applies to weak preference: if the businessman decides that, in each possible outcome of the election, buying is at least as good as not buying, then, without knowing the outcome, he can conclude that buying is at least as good as not buying.

The force of STP does not depend on there being only two options. It would apply equally well if there were many; say, he could buy or not buy any of a number of lots. If, under either of the assumptions, that the Republican will win and that the Democrat will win, buying Lot 1 is preferable (or weakly preferable) to buying Lot 2, then, without any assumption, buying Lot 1 is preferable (or weakly preferable) to buying Lot 2. STP concerns two acts under explicitly stated assumptions; its force derives only from these explicit assumption, and it remains the same, whatever properties the preference relation has, be it a total ordering, as Savage assumes, or only a partial one as is the case in other systems developed in the last forty years.

Let $\leqslant$ be the weak preference relation (of a rational agent): $f \leqslant g$ means that g is at least as good as f.[4] The strong ordering (g is better than f) can be then defined by: $f < g \Leftrightarrow_{\mathrm{Df}} f \leqslant g$ and $g \nleqslant f$. Savage's first postulate, P1, states that this is what he calls a "simple ordering" and which in current terminology is known as a total preorder: it is a reflexive transitive relation, and its being total means that for every f and g, either $f \leqslant g$ or $g \leqslant f$. In a preorder the conjunction $f \leqslant g$ & $g \leqslant f$ defines an equivalence relation, $\equiv$, which need not be the identity relation (as it is in total orderings); we get an induced ordering of the equivalence classes. Since different acts can be equi-preferable, "ordering" in the context of preference means a preorder, and this is the way I shall use "ordering" in the paper. That the ordering is total means that for any f, g exactly one of the three possibilities holds: either $f < g$, or $g < f$, or $f \equiv g$. In the last case we say that the agent is *indifferent* between f and g. Savage, who adopts STP without reservations, prefers however not to introduce it directly, since it would require additional machinery not included in his system (I shall return to this point shortly). He uses it rather as a guiding principle that leads him to his second postulate P2.

P2, and other postulates, can be stated more transparently than in Savage's book, by using partial acts: Let S be the set of all states, then a *partial act* is a function, r, defined over some event $E \subseteq S$, $E = \mathrm{Dom}(r)$, which assigns outcomes to all states in $\mathrm{Dom}(r)$ and is undefined otherwise.[5] An "act" in Savage's terminology (which is also the current adopted terminology) is *total*: it is a partial act whose domain is S. For a (total) act f and an event B, let $f|B =$ *the restriction of* f to B; it is the partial act with domain B, such that $f|B\,(s) = f(s)$, for all $s \in B$. Obviously, $f|B \cup h|B'$ is a partial act iff f and h agree on $B \cap B'$. In particular it is a partial act if $B \cap B' = \emptyset$; In that case let us rewrite $f|B \cup h|B'$ as:

$$f|B + h|B'$$

We shall use this notation as implying that $B \cap B' = \emptyset$. $f|B$ can be further restricted to C, where $C \subseteq B$; obviously $(f|B)|C = f|C$.[6] Unless stated otherwise, 'f', 'g', ... range over total acts. Hence '$f|A$', '$f|B$', ..., '$g|A$', '$g|B$', ... range over partial acts. Let $\overline{B} = S - B =$ the complement of B.[7]

The setup presupposes some non-empty set of total acts, call it Act. Savage assumes implicitly *closure under cut-and-paste*:

- If $f, g \in \mathrm{Act}$ and B is any event, then $f|B + g|\overline{B} \in \mathrm{Act}$.

The second postulate can be now stated as follows:

P2 For every event B: $f|B + h|\overline{B} \leqslant g|B + h|\overline{B} \Rightarrow f|B + h'|\overline{B} \leqslant g|B + h'|\overline{B}$

2 In possible-world lingo, the states are the possible worlds. They represent however only those aspects that are relevant to the problems at hand and, most importantly, the agent's choice of act is external to the possible world, not part of it.

3 Savage used bold face letters $\mathbf{f}, \mathbf{g}, \ldots$ for the acts, and $f, g, \ldots$ for the associated functions, although acts with the same associated functions are regarded equal. He also used 'f' 'g' ambiguously, to denote constant outcomes (the same in all states), in the cases where the act yields the "same" outcome in all states. We diverge from his notation also in other details.

4 Savage uses a special small-or-equal notation for acts; a special notation, such as '$\preccurlyeq$', is often used in the literature. I have preferred the simpler more economical choice. As long as we can read from the context the intended relation, we can help ourselves to '$\leqslant$'. The same considerations apply to the use of '+' below.

5 Partial acts are a technical device, and are used under the assumption that the state is in the act's domain. For more on partial acts and their comparison with other related kinds see [11].

6 Also other obvious equalities hold, for example: if $C \subseteq B$ then $(f|B + h|B')|C = f|C$.

7 In Savage's book this is denoted as $\sim B$.