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Ultralarge lotteries: Analyzing the Lottery Paradox using non-standard analysis

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ABSTRACT

A popular way to relate probabilistic information to binary rational beliefs is the Lockean Thesis, which is usually formalized in terms of thresholds. This approach seems far from satisfactory: the value of the thresholds is not well-specified and the Lottery Paradox shows that the model violates the Conjunction Principle. We argue that the Lottery Paradox is a symptom of a more fundamental and general problem, shared by all threshold-models that attempt to put an exact border on something that is intrinsically vague. We propose application of the language of relative analysis—a type of non-standard analysis—to formulate a new model for rational belief, called Stratified Belief. This contextualist model seems well-suited to deal with a concept of beliefs based on probabilities 'sufficiently close to unity' and satisfies a moderately weakened form of the Conjunction Principle. We also propose an adaptation of the model that is able to deal with beliefs that are less firm than 'almost certainty'. The adapted version is also of interest for the epistemicist account of vagueness.

There are no whole truths; all truths are half-truths. It is trying to treat them as whole truths that plays the devil.

[Alfred North Whitehead [39, p. 14]]

1. Introduction

In order to study the epistemology of yes–no beliefs, in particular the conditions for their rational acceptability, in so far as they are based on probabilistic information, we will focus on a simple example of a lottery. Consider a fair lottery with N tickets, exactly one of which will be randomly selected as the winner. This game of chance can simply be described by a uniform probability function, which assigns a probability of $\frac{1}{N}$ to each ticket. Clearly, the description of a fair lottery does not pose a problem within probability theory, but the interplay of probabilities and rational beliefs triggers epistemological questions. We are interested in what is rational to believe for a participant in such a lottery in the case that N is very large and the winning odds of a single ticket are correspondingly very small.

If you own only one ticket in such a large lottery, it may seem rational for you to believe that your ticket will not win. Buying another ticket does not increase your odds very much, so it still may seem rational for you to believe that none of your tickets will win. Suppose that you keep buying tickets, each with a very small probability of winning, and that you keep believing that none of your tickets will win. At some point, you will own all the lottery tickets and thereby you will

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be certain that one of them will win, which contradicts your belief that none of your tickets will win. This is the Lottery Paradox, originally stated by Kyburg [27].[1]

The Lottery Paradox occurs when three prima facie plausible principles are combined: the Lockean Thesis, the Conjunction Principle, and the Law of Non-Contradiction. This is the first principle:

Lockean Thesis (LT, Informal Version) It is rational to believe a statement if the probability of that statement is sufficiently close to unity.

The second principle, the Conjunction Principle or CP, states that if it is rational to believe two statements, it is also rational to believe their conjunction. The Law of Non-Contradiction or LNC expresses the idea that it is never rational to believe a contradiction. According to Kyburg himself, it is the employed aggregation rule for beliefs, CP, that causes the paradox [27]. Whereas Kyburg's argument that rational belief is not closed under conjunction was supported by Foley [15] and Klein [25], the idea that CP is the cause of the problem is now a minority position. Some doubt the Law of Non-Contradiction [40], but most contemporary authors are more suspicious of the Lockean Thesis. It has been suggested that LT be modified with a defeater-clause. It seems natural to assume that such a defeater can be made mathematically precise, but Douven and Williamson show that any formally precise defeater does not work to avoid the Lottery Paradox [13], reducing much of the initial appeal of this solution.

In this article, we will analyze the Lottery Paradox as an instantiation of vagueness. After all, the problem only occurs for a lottery ranging over a *large enough* number of tickets, making the probability of winning with a single ticket *small enough*. Also in the informal phrasing of LT, a vague element is present, where it states that the probability has to be *sufficiently close* to unity.

It is the goal of this paper to find a formal solution to the Lottery Paradox that does justice to this vagueness. It may not seem likely that a formal solution of this type exists: what mathematical method can help us out if the problem is intrinsically vague?

We propose to apply relative or stratified analysis [23,24], a type of non-standard analysis. Based on stratified analysis, we will give a formalization of LT and refer to the resulting type of rational belief as 'Stratified Belief'. As it turns out, CP will have to be adapted too, in order to be compatible with this soritic version of LT.

Regarding CP, the conclusion of this paper is close to the position of Kyburg, Foley, and Klein: we find that the Conjunction Principle is too strong to be expected to hold for rational beliefs. However, we do argue in favor of a weakened form of CP. So, like Kyburg, we claim that you would be wrong to keep believing that none of your tickets will win: the repeated addition of an extra ticket with a small probability does not guarantee that the total probability of all the tickets that you own remains small. The total probability of winning will be considerable before you have bought all the tickets. Yet, knowing this does not tell you exactly when you should stop adding tickets or change your opinion. The question "When do the winning odds of a number of tickets cease to be small?" is not all that different from "When does a number of lottery tickets start to be a heap?" In the application of the aggregation rule, we see that induction fails at some point, making the property of rational acceptability of beliefs intransitive.[2]

In the context of the philosophy of probability, two varieties of probability are considered: objective (or physical) probability on the one hand, and subjective (or epistemic) probability on the other. The probabilities occurring in physics are taken to be objective[3] and are thought of as real numbers in the $[0,1]$-interval. Subjective probabilities are often referred to as 'degrees of belief' in the Bayesian literature [42,16]. Unlike objective probabilities, degrees of belief do not necessarily have a numerical value. However, in the case of a lottery or other situations in which all relevant information about the objective probabilities is explicitly available, the agent's subjective probability assignments should be equal to the objective probabilities. This requirement has been dubbed "the Principal Principle" by Lewis [31] and we consider it as a minimal, necessary condition for rationality, underlying any attempt to formalize the notion of rational belief. Throughout this article, we will focus on the case in which the subjective probabilities are indeed equal to the objective ones. Alternatively, with Gaifman [17], we may consider a wide spectrum of forms of probability ranging from the purely objective to the purely subjective case, and observe that they all derive from a common kernel. In any case, for our present purposes it is sufficient that an agent assumes the lottery to be fair and reasons under this assumption. Therefore, we may use the term 'probability' without further qualification.

Notation Consider an N-ticket lottery, with N some natural number at least equal to 2. Here, we introduce some notation for probabilities of statements concerning such a lottery. Denote the set of N tickets as: $T_N = \{t_1, \ldots, t_N\}$. From this set, exactly one ticket will be randomly selected and assigned to be the winning ticket. If A is a subset of T_N, let $\varphi(A)$ denote the statement that one of the tickets in A is the winner. We introduce a similar notation for loss statements: if B is a subset of T_N, let $\psi(B)$ denote the statement that none of the tickets in B is the winner. Clearly, $\psi(B)$ is equivalent to $\varphi(T_N - B)$.

The assignment of probabilities (P) to win and loss statements can be done as follows:

[1] The paradox can be restated in terms of knowledge [35]. However, here we will address only the original phrasing in terms of rational belief.

[2] Intransitivity is a typical symptom of problems that are soritic in nature.

[3] Of course, even those probabilities are subjective to a certain extent: probability is a way to model a system about which we have insufficient information to predict its behavior with certainty or to summarize information about large numbers of particles.