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S7

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ABSTRACT

Following Halldén, we define S7 as the system generated by the addition of $\Diamond\Diamond p$ to S3. Initial motivation for the extension comes from Halldén's paradox. In addition to resolving the paradox, the resulting system generates a helpful framework for comparing classical propositional logic (CPL) with otherwise incommensurable logics, including multi-valued logics such as L3 and paraconsistent logics such as LP. S7, although non-regular and non-normal, thus turns out to be preferable to systems such as S4 and S5 as an account of alethic modality.

Following Halldén [9],[1] we define S7 as the system generated by the addition of the axiom

$$\Diamond\Diamond p \tag{1}$$

to S3. Initial motivation for the extension comes from Halldén's paradox. In addition to resolving the paradox, the resulting system gives a helpful framework for comparing classical propositional logic (CPL) with otherwise incommensurable logics, including multi-valued logics such as L3 and paraconsistent logics such as LP.[2] S7, although non-regular and non-normal, thus turns out to be preferable to systems such as S4 and S5 as an account of alethic modality.[3]

This paper comprises three main sections. Section 1 distinguishes S7 from more familiar systems such as T, B and S1 through S5. Section 2 makes explicit the comparison between S5 and S7. Section 3 discusses the benefits of S7 in the broader context of modality more generally.

1. We recall that T is the system that results from supplementing CPL with the operators $\Box$, $\Diamond$ and ⥽, together with *necessitation*, the rule of inference that

$$\text{If} \vdash \alpha, \quad \text{then} \vdash \Box\alpha, \tag{N}$$

and the axioms

$$\Box p \supset p \tag{2}$$

and

$$\Box(p \supset q) \supset (\Box p \supset \Box q). \tag{3}$$

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[1] Earlier discussion appears in Lewis and Langford [16], p. 497.

[2] For details concerning L3, see Malinowski [17]. For details concerning LP, see Priest [19], [20] and [22]. For helpful surveys of paraconsistency, see Priest [21], Priest, Routley and Norman [24] and Schotch, Brown and Jennings [27]. For helpful overviews of non-standard logics more generally, see Haack [8], Burgess [3], Gabbay and Woods [6] and [7].

[3] Alternative approaches include those of Mares [18], Varzi [29] and Gabbay, Kurucz, Wolter and Zakharyaschev [5].

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S1 is the result of replacing (N) with the weaker rule

$$\text{If } \vdash_{CPL} \alpha, \quad \text{then } \vdash_{S1} \Box\alpha. \qquad (N_{S1})$$

S2 is the result of supplementing S1 with the Consistency Postulate,

$$\Diamond(p \wedge q) \prec \Diamond p. \qquad (4)$$

S3 is the result of supplementing S1 with the stronger postulate

$$(p \prec q) \prec (\Box p \prec \Box q). \qquad (5)$$

B is the result of supplementing T with Brouwer's axiom,

$$p \supset \Box\Diamond p. \qquad (6)$$

S4 is the result of supplementing T with the axiom

$$\Box p \supset \Box\Box p, \qquad (7)$$

and S5 is the result of supplementing T with the axiom

$$\Diamond p \supset \Box\Diamond p.^{4} \qquad (8)$$

As has long been known, although incompatible with systems S4 and S5, axiom (1) is independent of each of the systems S1, S2 and S3. As a result, the addition of (1) to S2 has come to be known as S6,[5] the addition of (1) to S3 has come to be known as S7,[6] the addition of

$$\Box\Diamond\Diamond p \qquad (9)$$

to S3 as S8,[7] and the addition of (1) together with

$$\Diamond p \supset \Box\Diamond p \qquad (10)$$

to S3 as S9.[8] One way of understanding relations between these systems appears in Diagram 1. Systems to the left of the broken line and to the right of the double solid line neither contain nor are contained in S5. Systems to the right of the single solid line contain K and so *include* rule (N). Systems to the left of the single solid line *lack* rule (N). Systems to the left of the broken line are *inconsistent* with rule (N).

S7 turns out to be both non-regular and non-normal. We see this as follows[9]: Corresponding to any modal formula is its PC-transform, the formula obtained from the original formula by eliminating all occurrences of strict implication or strict equivalence through the use of definitions and equivalences and then deleting all occurrences of $\Box$ and $\Diamond$. A system is *regular* provided the PC-transforms of all its theorems turn out to be valid formulae in CPL. (Proving a system to be regular is thus one way of giving a consistency proof for that system.) Since S7 contains formulae of the form $\Diamond\Diamond p$, there is no way to guarantee that all its PC-transforms will be valid. Hence S7 is non-regular. It is also consistent[10] and complete.[11]

A system is *normal* provided it contains (2) and (3) as either axioms or theorems, as well as the transformation rules *modus ponens*

$$\text{If } \vdash \alpha, \text{ and } \vdash \alpha \supset \beta, \quad \text{then } \vdash \beta \qquad (MP)$$

and *necessitation*, or their equivalents.[12]

Systems such as S1–S3 and S6–S9 which lack (N) are thus non-normal, a result Kripke finds "intuitively somewhat unnatural."[13] Halldén goes a step further, noting that since systems such as S7 not only lack, but are inconsistent with, (N), "No interesting interpretation of the calculi S6–S8 is known."[14]

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4 T, S4 and S5 are also known respectively as M, M′ and M″. Terminology here follows that of Hughes and Cresswell. See Hughes and Cresswell [11], pp. 17–18, 30–31, 223-5, 230, 233, 257, 46 and 49.

5 Alban [1], p. 25.

6 Halldén [9], p. 230.

7 Halldén [9], p. 230.

8 Aqvist [2], pp. 79, 82. Named S7.5 by Aqvist, the system was later proved to contain S8. As a result, it was renamed S9 by Hughes and Cresswell. See Hughes and Cresswell [11], p. 272.

9 Discussion appears in Halldén [9] and [10], Sobocinski [28], and Hughes and Cresswell [11], pp. 267ff.

10 Halldén [9], p. 230.

11 Cresswell [4].

12 Although the definition is due to Kripke, terminology here follows that of Hughes and Cresswell. (See Hughes and Cresswell [11], pp. 31, 237.) Being a rule whose antecedent refers only to theorems, necessitation is distinct from the invalid formula $p \supset \Box p$, which of course would be incompatible with all but the most deterministic of worlds. See Kripke [14], p. 67. Some authors prefer the formulation, If $\models \alpha$, then $\models \Box\alpha$.

13 Kripke [15], p. 206.

14 Halldén [9], p. 230. Kielkopf [12] goes even further, claiming that (1) entails nihilism.