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A sequent calculus for a logic of contingencies

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ABSTRACT

We introduce a sequent calculus that is sound and complete with respect to propositional *contingencies*, i.e., formulas which are neither provable nor refutable. Like many other sequent and natural deduction proof systems, this calculus possesses cut elimination and the subformula property and has a simple proof search mechanism.

1. Introduction

We are frequently interested in *non-trivial* statements, i.e., statements which are true in some possible worlds, and false in others. In classical propositional logic, such *contingent* statements are easily defined *semantically*, e.g., by non-constant truth assignments. Proving contingency of a statement using exhaustive search over all propositional interpretations is possible for small-size formulas, but for real problems we would like to have something similar to a theorem prover — a “contingency prover”.[1]

For the first time, a contingency proof system appeared in [8], and the first proper proof system was recently introduced in [6]. The latter system is based on transformation rules allowing the user to translate a formula into a perfect disjunctive normal form, and then to remove the redundant variables in order to obtain a contingency axiom.

In this paper we introduce a contingency sequent calculus that, like other sequent calculi, seems to be more appropriate for automatic theorem proving. Of course, our calculus cannot solve the complexity of the decision problem for contingencies, because it is trivially NP-complete. However, in some cases, the sequent prover might produce a shorter proof. In addition, the contingency sequent calculus is much simpler than other proof systems, and better represents the nature of contingency. This is because simplicity is a common feature of sequent and natural deduction calculi versus Hilbert like proof systems.

This paper is organized as follows. In the next section we present syntax and semantics of contingency. Section 3 deals with the propositional contingency calculus. Namely, this section contains the proofs of soundness, completeness, cut elimination, and the subformula property for contingency calculus. Finally, in Section 4 we summarize the main results of this paper and present a direction for future research.

2. Syntax and semantics

The underlying language is the language of propositional logic with standard propositional connectives $\vee$ (disjunction), $\wedge$ (conjunction), $\supset$ (implication), and $\neg$ (negation) and an infinite set of propositional variables *Var*.

[1] A similar problem exists for unprovable and satisfiable formulas, where various proof systems were proposed for propositional logic and for the first-order logic of finite interpretations, see [1,2,4,5,7,8].

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A *sequent* is an expression of the form $\Gamma \to \Delta$, where Γ and Δ are finite (possibly empty) sequences of formulas. To define the propositional counterpart of a sequent we need the following notation. For a sequence of formulas $\Theta = A_1, A_2, \ldots, A_n$, we denote by $\bigwedge \Theta$ and $\bigvee \Theta$ the formulas $\bigwedge_{i=1}^{n} A_i$ and $\bigvee_{i=1}^{n} A_i$, respectively.[2] In this notation, the propositional counterpart of the sequent $\Gamma \to \Delta$ is the formula $\bigwedge \Gamma \supset \bigvee \Delta$. A sequent $\Gamma \to \Delta$ is *contingency* if the formula $\bigwedge \Gamma \supset \bigvee \Delta$ is neither true nor false.

Next we recall the semantics of sequents. An interpretation $\mathfrak{M}$ is a set of propositional variables and satisfiability of a formula φ by $\mathfrak{M}$, denoted $\mathfrak{M} \models \varphi$, is defined in the usual manner. We say that $\mathfrak{M}$ satisfies a sequent $\Gamma \to \Delta$, denoted $\mathfrak{M} \models \Gamma \to \Delta$, if $\mathfrak{M} \models \varphi$ for some formula φ that occurs in Δ, or $\mathfrak{M} \not\models \psi$ for some formula ψ that occurs in Γ. That is, $\mathfrak{M} \models \Gamma \to \Delta$ if $\mathfrak{M} \models \bigwedge \Gamma \supset \bigvee \Delta$.

Semantics of contingency is based on two interpretations which serve as *contingency witnesses*: one for establishing that a formula/sequent is not falsity, and the other for establishing that it is not a tautology.

We say that the pair of interpretations $(\mathfrak{M}, \mathfrak{N})$ is a *contingency witness* for a formula φ, denoted $(\mathfrak{M}, \mathfrak{N}) \models_c \varphi$, if $\mathfrak{M} \models \varphi$, and $\mathfrak{N} \not\models \varphi$. The formula φ is *contingent*, denoted $\models_c \varphi$, if it has a contingency witness. Note that $\models_c \varphi$ if and only if $\models_c \neg\varphi$.

The pair of interpretations $(\mathfrak{M}, \mathfrak{N})$ is a contingency witness for a sequent $\Gamma \to \Delta$, denoted $(\mathfrak{M}, \mathfrak{N}) \models_c \Gamma \to \Delta$, if $\mathfrak{M} \models \Gamma \to \Delta$ and $\mathfrak{N} \not\models \Gamma \to \Delta$. The sequent $\Gamma \to \Delta$ is *contingently valid*, or just *c*-valid, denoted $\models_c \Gamma \to \Delta$ if it has a contingency witness, i.e., $(\mathfrak{M}, \mathfrak{N}) \models_c \Gamma \to \Delta$ for some pair of propositional interpretations $\mathfrak{M}$ and $\mathfrak{N}$.

3. Propositional contingency calculus

The proof system for propositional contingencies is similar to that in [1,7], where sequent proof systems for non-tautologies (anti-sequents) were introduced. Naturally, the set of axioms for contingencies is a subset of the set of axioms for non-tautologies. However, the introduction rules for $\vee$ and $\supset$ into antecedent and the introduction rules for $\wedge$ into succedent are very different from the corresponding rules for anti-sequents. Note that contingency is preserved under negation, and therefore, it is natural that many rules move a formula from the antecedent of a sequent to its succedent and vice versa.

3.1. Propositional axioms and introduction rules

The *Contingency Calculus* (CC) is defined as follows. It has one axiom scheme $\boldsymbol{Ax_{CC}}$ that consists of all sequents $\Gamma \to \Delta$ such that

- Γ and Δ are (finite) sequences of propositional variables, i.e., elements of *Var*,
- Γ is not empty or Δ is not empty, and
- Γ and Δ have no common elements, i.e., no propositional variable occurs both in Γ and Δ.

Note that $\boldsymbol{Ax_{CC}}$ is similar to the propositional part of the scheme in [7]. The only difference is that $\boldsymbol{Ax_{CC}}$ does not contain the empty sequent, because this sequent is false in every interpretation.

Usually, for every logical connective, a sequent calculus has introduction rules into both antecedent and succedent. This also happens for the contingency sequent calculus CC. However, the semantics of contingency is different from that of validity, because in many rules the main formula is actually moved to the opposite part of the sequent. Therefore, some of the rules of CC have no counterparts in the classical standard sequent calculus like ***LK*** from [3, Chapter III, § 1], or even in the contingency sequent calculus ([1,7]).

The introduction rules for the logical connectives are as follows.[3]

$$\neg \quad \frac{\Gamma \to \Delta, \varphi}{\Gamma, \neg\varphi \to \Delta} \qquad \frac{\Gamma, \varphi \to \Delta}{\Gamma \to \Delta, \neg\varphi}$$

$$\wedge \quad \frac{\Gamma, \varphi, \psi \to \Delta}{\Gamma, \varphi \wedge \psi \to \Delta} \qquad \frac{\Gamma, \varphi, \psi \to \Delta \qquad \Gamma \to \Delta, \varphi}{\Gamma \to \Delta, \varphi \wedge \psi} \qquad \frac{\Gamma, \varphi, \psi \to \Delta \qquad \Gamma \to \Delta, \psi}{\Gamma \to \Delta, \varphi \wedge \psi}$$

$$\vee \quad \frac{\Gamma \to \Delta, \varphi, \psi \qquad \Gamma, \varphi \to \Delta}{\Gamma, \varphi \vee \psi \to \Delta} \qquad \frac{\Gamma \to \Delta, \varphi, \psi \qquad \Gamma, \psi \to \Delta}{\Gamma, \varphi \vee \psi \to \Delta} \qquad \frac{\Gamma \to \Delta, \varphi, \psi}{\Gamma \to \Delta, \varphi \vee \psi}$$

$$\supset \quad \frac{\Gamma, \varphi \to \Delta, \psi \qquad \Gamma \to \Delta, \varphi}{\Gamma, \varphi \supset \psi \to \Delta} \qquad \frac{\Gamma, \varphi \to \Delta, \psi \qquad \Gamma, \psi \to \Delta}{\Gamma, \varphi \supset \psi \to \Delta} \qquad \frac{\Gamma, \varphi \to \Delta, \psi}{\Gamma \to \Delta, \varphi \supset \psi}$$

There are also six *structural* rules of inference in CC. The first two are the classical *interchange* rules

$$\textit{Interchange} \quad \frac{\Gamma', \varphi, \psi, \Gamma'' \to \Delta}{\Gamma', \psi, \varphi, \Gamma'' \to \Delta} \qquad \frac{\Gamma \to \Delta', \varphi, \psi, \Delta''}{\Gamma \to \Delta', \psi, \varphi, \Delta''}$$

[2] Thus, if $n = 0$, i.e., Θ is empty, $\bigwedge \Theta$ is *truth* and $\bigvee \Theta$ is *falsity*.

[3] Note the duality between $\vee$ and $\wedge$, and between introductions of $\neg$ into antecedent and succedent.