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A note on orthogonality of subspaces in Euclidean geometry

Jacek Konarzewski, Mariusz Żynel *

Institute of Mathematics, University of Białystok, Akademicka 2, 15-267 Białystok, Poland

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ABSTRACT

We show that Euclidean geometry in suitably high dimension can be expressed as a theory of orthogonality of subspaces with fixed dimensions and fixed dimension of their meet.

1. Introduction

While the notion of orthogonality of lines in Euclidean geometry has well founded meaning (it is frequently used as a primitive notion, see [3,4]), orthogonality of subspaces can be defined in several different ways. Two of them were shown in [5] to be sufficient in Euclidean geometry; actually, each of these two considered on the universe of subspaces of fixed dimension can be used to reinterpret the underlying point-line affine space and after that to define line orthogonality. Thus the procedure of reinterpretation consists, in fact, in two steps and in the second step one should define orthogonality of lines in terms of a given orthogonality of subspaces. In this note we show that such a definition is possible for each prescribed values of dimensions of the considered subspaces (Theorem 2.4(ii)).

The notion of orthogonality of subspaces is not a unique-meaning relation, even if dimensions of the subspaces involved are fixed. Therefore, we have to deal with a family of possible relations of orthogonality. And in this note we show that *each one* of these relations is sufficient to express the underlying geometry provided the latter has sufficiently high dimension (Theorem 2.4(i)).

So, finally, we prove that Euclidean geometry can be expressed in the language with points, subspaces (of fixed dimensions), and orthogonality of subspaces. It is a folklore that affine geometry can be expressed as a theory of point-k-subspace incidence. Euclidean geometry appears when we impose a relation of orthogonality on that "affine" structure.

Our result does not solve the problem whether Euclidean geometry can be expressed in the language with k-subspaces *as individuals* and some of the orthogonalities introduced above as a single primitive notion, in that way, possibly, generalizing [5]. We conjecture that the answer is affirmative, but the question is addressed in other papers.

We close the paper with a list of some more interesting properties of the orthogonalities considered here. This list is not intended as a complete axiom system, but we think that at least some of its items can be used to build such a system characterizing orthogonality of subspaces.

2. Results

Let $\mathfrak{M} = \langle S, \mathcal{L}, \perp\rangle$ be an Euclidean space, where $\mathfrak{A} := \langle S, \mathcal{L}\rangle$ is an affine space with $\mathcal{L} \subset 2^S$ and $\perp \subset \mathcal{L} \times \mathcal{L}$ is a line orthogonality (cf. [3]). Up to an isomorphism $\mathfrak{M}$ corresponds to $\langle V, \mathcal{L}_V, \perp_\xi\rangle$ where V is a vector space, $\mathcal{L}_V$ is the set of

* Corresponding author.
E-mail addresses: konarzewski20@wp.pl (J. Konarzewski), mariusz@math.uwb.edu.pl (M. Żynel).

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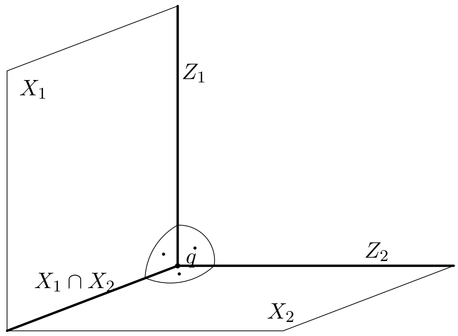

Fig. 1.

translates of 1-dimensional subspaces of V and $\perp_\xi$ is the orthogonality determined by a nondegenerate symmetric bilinear form ξ on V with no isotropic directions. For each nonnegative integer k, $\mathcal{H}_k$ stands for the class of all k-dimensional subspaces of $\mathfrak{M}$, and $\mathcal{H}$ stands for all subspaces of $\mathfrak{M}$. The empty set $\emptyset$ is considered an affine subspace, that is $\emptyset \in \mathcal{H}$. If $X_1, X_2 \in \mathcal{H}$ we write $X_1 \sqcup X_2$ for the least subspace in $\mathcal{H}$ that contains $X_1 \cup X_2$ (i.e. the meet of all elements of $\mathcal{H}$ containing $X_1 \cup X_2$). Note that the ground field is not GF(2). So, from elementary affine geometry we have this fact.

Fact 2.1.

(i) *The family $\mathcal{H}_k$ is definable in $\mathfrak{A}$ for each nonnegative integer $k \leqslant \dim(\mathfrak{A})$.*
(ii) *Let $0 < k < \dim(\mathfrak{A})$. Then the family $\mathcal{L}$ is definable in the incidence structure $\langle S, \mathcal{H}_k\rangle$. Consequently, $\mathfrak{A}$ is definable in $\langle S, \mathcal{H}_k\rangle$.*

Recall that $\mathfrak{M}$ is definitionally equivalent to the structure $\langle S, \mathcal{L}, \perp\!\!\!\perp\rangle$ (cf. e.g. an axiom system for $\perp\!\!\!\perp$ in [2,7]), where $\perp\!\!\!\perp \subset S^2 \times S^2$ is defined in $\mathfrak{M}$ by the formula

$$a, b \perp\!\!\!\perp c, d \quad :\Longleftrightarrow \quad \text{there are } L_1, L_2 \in \mathcal{L} \text{ such that } a, b \in L_1 \perp L_2 \ni c, d. \tag{1}$$

Given any two $X, Y \in \mathcal{H}$ we write

$$X \perp Y \quad :\Longleftrightarrow \quad a, b \perp\!\!\!\perp c, d \quad \text{for all } a, b \in X,\ c, d \in Y. \tag{2}$$

Note that for $X, Y \in \mathcal{L}$ the relation defined by (2) coincides with the orthogonality we have started from. If $X \perp Y$ then $X \cap Y$ is at most a point; we write

$$X \perp^* Y \quad :\Longleftrightarrow \quad X \perp Y \quad \text{and} \quad X \cap Y \neq \emptyset. \tag{3}$$

Recall that for any two subspaces $X, V \in \mathcal{H}$ such that $X \subset V$ and any point $q \in X$ there is the unique maximal $X' \in \mathcal{H}$ such that $q \in X' \perp^* X$, $X' \subset V$, and $X \sqcup X' = V$. We call X' an *orthocomplement* of X in V through q. If X is a point then necessarily $X = \{q\}$ and $X' = V$.

Let us define now (cf. Fig. 1)

$$\begin{aligned} X_1 \mathrel{\underline{\phi}} X_2 \quad :\Longleftrightarrow \quad & \text{there is a point } q \in X_1 \cap X_2 \text{ and } Z_1, Z_2 \in \mathcal{H} \text{ such that} \\ & q \in Z_1, Z_2 \perp^* X_1 \cap X_2, \quad Z_1 \perp^* Z_2 \quad \text{and} \quad (X_1 \cap X_2) \sqcup Z_i = X_i \quad \text{for } i = 1, 2. \end{aligned} \tag{4}$$

It is seen that the relation $\mathrel{\underline{\phi}}$ is symmetric. It is also not too hard to note that the following holds

$$X_1 \mathrel{\underline{\phi}} X_2 \quad \Longleftrightarrow \quad \text{there is } Z_i \in \mathcal{H} \text{ such that } Z_i \perp^* X_{3-i} \text{ and } (X_1 \cap X_2) \sqcup Z_i = X_i \tag{5}$$

for both $i = 1, 2$. Note that when $X_1 \cap X_2$ is a point then $X_1 \mathrel{\underline{\phi}} X_2$ and $X_1 \perp^* X_2$ are equivalent. Recall also a known formula

$$q \in X_1, X_2 \perp^* Y \ni q \quad \Longrightarrow \quad Y \perp^* (X_1 \sqcup X_2). \tag{6}$$

The motivation for such general definition (4) is reflection geometry (cf. [1,6]). It is known that the commutativity of reflections on subspaces characterizes orthogonality or incidence. More precisely: two reflections on different subspaces commutate iff the subspaces are orthogonal or incident. In our settings, denote by σ_X the reflection in a subspace X, i.e. an involutory isometry that fixes X pointwise; then

$$\sigma_{X_1}\sigma_{X_2} = \sigma_{X_2}\sigma_{X_1} \iff X_1 \mathrel{\underline{\phi}} X_2. \tag{7}$$

One might call $\mathrel{\underline{\phi}}$ an orthogonality, but note that (7) yields the formula

$$X_1 \subset X_2 \quad \Longrightarrow \quad X_1 \mathrel{\underline{\phi}} X_2, \tag{8}$$