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Representation of interlaced trilattices

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ABSTRACT

Trilattices are algebraic structures introduced ten years ago into logic with the aim to provide a uniform framework for the notions of constructive truth and constructive falsity. In more recent years, trilattices have been used to introduce a number of many-valued systems that generalize the Belnap–Dunn logic of first-degree entailment, proposed as logics of how several computers connected together in a network should think in order to deal with incomplete and possibly contradictory information. The aim of the present work is to develop a first purely algebraic study of trilattices, focusing in particular on the problem of representing certain subclasses of trilattices as special products of bilattices. This approach allows to extend the known representation results for interlaced bilattices to the setting of trilattices and to reduce many algebraic problems concerning these new structures to the better-known framework of lattice theory.

1. Introduction

Trilattices were first introduced into logic by Y. Shramko, J.M. Dunn and T. Takenaka [21] with the aim to provide a uniform framework for the notions of constructive truth and constructive falsity. These algebraic structures were used to define some interesting many-valued logics that Shramko and his collaborators proposed as generalizations of the systems introduced by A. Heyting as a formal counterpart of constructive (intuitionistic) logic and by D. Nelson [15] as a logic for constructive falsity.

Logics based on trilattices are also closely related to other well-known formal systems such as bilattice and relevance logics. This relationship has been stressed and investigated in several works by Y. Shramko and H. Wansing [19,20,22], who presented their trilattice logics as a generalization of the "useful four-valued logic" introduced by N. Belnap and J.M. Dunn [3,1]. While the Belnap–Dunn system was originally proposed as a logic of how a computer should think in order to handle information coming from different and possibly conflicting sources, Shramko and Wansing proposed trilattice-based systems as logics meant to model how several computers connected together in a network should think in order to deal with incomplete and possibly contradictory information.

The aim of the present work is to provide a first algebraic approach to the study of trilattices, focusing in particular on the relationship between trilattices and bilattices, in order to extend some of the representation results obtained in [4] for bilattices to the setting of trilattices. The main appeal of this approach, that proved to be useful in the case of bilattices, is that it allows to reduce many algebraic problems concerning these new structures to the better-known framework of lattices, in which they can be solved using powerful tools and results of lattice theory.

The paper is organized as follows. The next section contains the main definitions and fixes the terminology that we are going to use; it presents as well some basic results on bilattices and trilattices that we shall need in the subsequent sections. Section 3 contains some of the main results of this paper, namely representation theorems stating that various kinds of trilattices can be constructed as special products of two bilattices. At the end of Section 3.5 we briefly compare

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our approach to a previous work by S. Odintsov on the representation of a particular example of trilattice. In Section 4 we use the representation results of Section 3 to obtain characterizations of the congruences of trilattices in terms of those of their bilattice factors. These results are then used in Section 5 in order to identify the generators of minimal varieties of trilattices (i.e., the distributive ones). Finally, Section 6 mentions some open problems and lines for future research.

2. Definitions and basic results

In this section we introduce the main definitions, terminology and notation that we are going to use throughout the present work.

2.1. Bilattices

A *pre-bilattice* [8] is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ such that $\langle B, \leqslant, \wedge, \vee \rangle$ and $\langle B, \sqsubseteq, \sqcap, \sqcup \rangle$ are both lattices. For notational convenience, we shall sometimes indicate the pre-bilattice $\langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ just as $\langle B, \leqslant, \sqsubseteq \rangle$, but let us stress that we always treat these structures as algebras (rather than as doubly partially ordered sets).

In the literature on bilattices it is usually required that both lattices be complete or at least bounded, but here none of these assumptions is made. The minimum and maximum element of the lattice $\langle B, \wedge, \vee \rangle$, in case they exist, will be denoted, respectively, by f and t. Similarly, $\bot$ and $\top$ will refer to the minimum and maximum of $\langle B, \sqcap, \sqcup \rangle$, when they exist.

In logical contexts, where the underlying set of a pre-bilattice is understood as a space of truth values, the two lattice orders are usually thought of as representing the degree of truth ($\leqslant$) and the degree of information ($\sqsubseteq$) associated with a given sentence; accordingly, they are called respectively the *truth order* (or "logical order") and the *information order* (or "knowledge order"). This accounts for the use of f (for *false*) and t for (*true*) to denote the least and greatest elements w.r.t. the truth order, while $\bot$ should represent a complete absence of information and $\top$ an excess of it (a contradiction).

We reserve the term *bilattice* [12] for what is sometimes called a "bilattice with negation", i.e., an algebra $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ such that $\langle B, \wedge, \vee, \sqcap, \sqcup \rangle$ is a pre-bilattice and the negation $\neg : B \to B$ is an operation satisfying that, for all $a, b \in B$:

$$\text{if } a \leqslant b, \quad \text{then } \neg b \leqslant \neg a$$
$$\text{if } a \sqsubseteq b, \quad \text{then } \neg a \sqsubseteq \neg b$$
$$a = \neg\neg a.$$

Negation is thus anti-monotonic with respect to the truth order and monotonic with respect to the information order; it is not difficult to convince oneself that these requirements constitute a plausible generalization of the behavior of negation within classical logic. The following identities (that we will call *De Morgan laws*) hold in any bilattice:

$$\neg(a \wedge b) = \neg a \vee \neg b \qquad \neg(a \vee b) = \neg a \wedge \neg b$$
$$\neg(a \sqcap b) = \neg a \sqcap \neg b \qquad \neg(a \sqcup b) = \neg a \sqcup \neg b.$$

Moreover, if the bilattice is bounded, then $\neg\top = \top$, $\neg\bot = \bot$, $\neg\mathsf{t} = \mathsf{f}$ and $\neg\mathsf{f} = \mathsf{t}$. So, if a bilattice $\mathbf{B} = \langle B, \wedge, \vee, \sqcap, \sqcup, \neg \rangle$ is distributive, or at least the reduct $\langle B, \wedge, \vee \rangle$ is distributive, then $\langle B, \wedge, \vee, \neg \rangle$ is a De Morgan lattice.

The most interesting algebraic results known on (pre-)bilattices, in particular the representation theorems that we are going to state below, do not apply to all bilattices, but only to the subclass of the interlaced ones (most of these results may be found in [4,5], to which we refer for more details and the proofs that we are going to omit).

A pre-bilattice is called *interlaced* [7] when all four lattice operations are monotone w.r.t. to both lattice orders. It is called *distributive* [12] when all possible distributive laws concerning the four lattice operations, i.e., any identity of the following form, hold:

$$a \circ (b \bullet c) = (a \circ b) \bullet (a \circ c) \quad \text{for every } \circ, \bullet \in \{\wedge, \vee, \sqcap, \sqcup\}.$$

We say that a bilattice is interlaced (or distributive) when its pre-bilattice reduct is.

Fig. 1 shows the double Hasse diagram of some of the best-known (pre-)bilattices: the four- and nine-element ones are distributive, while the seven-element one is not (in fact, it is not even interlaced). The diagrams should be read as follows: $a \leqslant b$ if there is a path from a to b which goes uniformly from left to right, while $a \sqsubseteq b$ if there is a path from a to b which goes uniformly from the bottom to the top. The four lattice operations are thus uniquely determined by the diagram, while negation, if there is one, corresponds to reflection along the vertical axis joining $\bot$ and $\top$. It is then clear that all the pre-bilattices shown in Fig. 1 can be endowed with a negation in a unique way and turned in this way into bilattices. When no confusion is likely to arise, we will use the same name to denote a particular pre-bilattice and its associated bilattice. The names used in the diagrams are by now more or less standard in the literature, except for the subscripts, that we use to indicate that we are now considering structures endowed with two lattice orders (whereas further below we shall consider three orders).

The smallest non-trivial bilattice, $\mathcal{FOUR}_2$, has a fundamental role among bilattices, both from an algebraic and a logical point of view. $\mathcal{FOUR}_2$ is distributive and, as a bilattice, it is a simple algebra. It is in fact, up to isomorphism, the only