Journal of Applied Logic 11 (2013) 190–216

Contents lists available at SciVerse ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

Modal definability of first-order formulas with free variables and query answering

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ARTICLE INFO

Article history:
Received 4 July 2012
Accepted 13 March 2013
Available online 20 March 2013

Keywords:
Modal logic
Modal definability
Correspondence theory
Description logic
Knowledge base
Conjunctive query

ABSTRACT

We present an algorithmically efficient criterion of modal definability for first-order existential conjunctive formulas with several free variables. Then we apply it to establish modal definability of some family of first-order ∀∃-formulas. Finally, we use our definability results to show that, in any expressive description logic, the problem of answering modally definable conjunctive queries is polynomially reducible to the problem of knowledge base consistency.

1. Introduction

The correspondence between modal and first-order (FO) formulas on Kripke frames is the heart of modal logic. Developed in the 1960s, it is still a common tool for establishing completeness of many modal calculi. A typical modern example of its application is given by various logics of multi-agent systems for reasoning about agents' knowledge, belief, intentions, and cooperative actions [10].

Traditionally, two kinds of correspondence are studied: the *global* one between modal formulas and closed FO formulas, and the *local* one between modal formulas and FO formulas with one free variable. It was Kracht who first introduced in [23] the notion of correspondence between *n-tuples* of modal formulas and FO formulas with n free variables, for arbitrary $n \geqslant 1$. He established basic properties of this notion and devised a special calculus (called "the calculus of internal descriptions") for deriving instances of such a correspondence. In [23] he used this notion of correspondence for proving the claim, known now as Kracht's theorem [3,22], which describes a large class of FO formulas that are modally definable.

Typically, this notion of correspondence is used only as a technical tool for proving similar theorems (see, e.g., [18]). However, recently a query answering algorithm based on the local correspondence emerged [36,37]. Its key idea is to replace a query (which is a FO formula) with a corresponding modal formula. Since this algorithm is based on the *local* correspondence, the range of its applications is limited to unary queries, i.e., to FO formulas with one free variable. Now, this limitation can be overcome by considering a more general kind of correspondence, i.e., modal definability of FO formulas with several free variables. This is the departing point for our research.

In this paper, we mainly focus on modal definability of FO formulas of a special kind, called *existential conjunctive formulas*, or ∃&-formulas, for short. An ∃&-formula is an existentially quantified conjunction of atomic formulas of the form $x\mathsf{R}y$;

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1570-8683/$ – see front matter
http://dx.doi.org/10.1016/j.jal.2013.03.007

for instance, $\exists y\,(x\mathsf{R}y \wedge y\mathsf{R}y)$. The motivation for considering these formulas is not only that they form a natural fragment of FO logic, but also that they are closely related to so-called *conjunctive queries*, which play an important rôle in knowledge representation and reasoning.

The main result of our paper is the algorithmically efficient criterion of modal definability for ∃&-formulas with several free variables. Moreover, given a modally definable ∃&-formula, our algorithm produces, in polynomial time, the corresponding tuple of modal formulas. This contrasts to the general case, for it is undecidable whether an arbitrary FO formula (even with one free variable) is modally definable, due to Chagrova's result [7].

The paper is organized as follows. Section 2 recalls the notion of modal definability (and introduces its generalization, modal expressibility) for FO formulas with several free variables. Section 3 introduces the family of ∃&-formulas, together with their graph representation. Section 3.1 presents our main result — the criterion of modal definability of ∃&-formulas, formulated in graph-theoretic terms. The proofs of definability and undefinability results are given in Sections 4 and 5, respectively. In Section 6 we use our results to prove modal definability of a large family of ∀∃-formulas, which arise in many-dimensional modal logics.

The last part of our paper, Section 7, gives an application of our definability results to the problem of answering conjunctive queries in description logic knowledge bases. There we recall all necessary definitions and cornerstone results in that field. Then, the Reduction Theorem (Theorem 7.2) establishes the relationship between modal definability and query answering in FO theories. Its consequence (Theorem 7.8) says that for modally definable conjunctive queries, the problem of query answering can be solved easier than that for arbitrary conjunctive queries, by a polynomial reduction to the problem of knowledge base consistency. The concluding Section 8 points out directions for further research.

2. Modal definability

Modal formulas are built up from propositional variables $\mathsf{PV} = \{p_0, p_1, \ldots\}$ and modal operators $\{\Box_\ell \mid \ell \in L\}$ according to the following syntax:

$$\varphi, \psi ::= p_i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box_\ell\varphi.$$

Other connectives are taken as standard abbreviations, e.g. $\Diamond_\ell\varphi = \neg\Box_\ell\neg\varphi$.

Let us recall the Kripke semantics. A *frame* $F = (W, (R_\ell)_{\ell\in L})$ consists of a nonempty set W of *points* or *worlds* and binary relations $R_\ell \subseteq W \times W$. A *model* is a pair $M = (F, \theta)$, where F is a frame and θ a *valuation* on F, i.e., a function that assigns to each variable p a set $\theta(p) \subseteq W$. The *truth* of a formula φ at a point $w \in W$ in a model M (denoted by $M, w \models \varphi$ or $F, \theta, w \models \varphi$) is defined in a standard way. In particular, $M, w \models \Box_\ell\varphi$ iff $M, u \models \varphi$ for all points $u \in W$ with $wR_\ell u$. A formula φ is called *valid* at a point w of a frame F (notation: $F, w \models \varphi$) if $F, \theta, w \models \varphi$ for all valuations θ.

Next fix a countable set Var of individual variables and consider the first-order (FO) language with equality in the signature of binary relation symbols[1] R_ℓ, for each $\ell \in L$. Observe that a frame F can serve as an interpretation for this language. Hence, given a FO formula $A(x_1, \ldots, x_n)$ with n free variables and n elements $e_1, \ldots, e_n \in W$, the relation $F \models A(e_1, \ldots, e_n)$ is well-defined.

Now we come to the central definition of our paper, first proposed by Kracht in [23]. Intuitively, it formalizes the notion of a first-order formula $A(x_1, \ldots, x_n)$ and a tuple of modal formulas $\langle\varphi_1, \ldots, \varphi_n\rangle$ being equivalent in some sense. Unless otherwise stated, below we assume that $n \geqslant 1$, so that we do not consider closed FO formulas.

Definition 2.1. A FO formula $A(x_1, \ldots, x_n)$ *corresponds* to a tuple of modal formulas $\langle\varphi_1, \ldots, \varphi_n\rangle$ if, for any frame F and any points $e_1, \ldots, e_n$ in F, the equivalence holds:

$$F \models A(e_1, \ldots, e_n) \iff \text{for every valuation } \theta \text{ there is } i \leqslant n \text{ with } F, \theta, e_i \models \varphi_i.$$

In this case we write $A(\vec{x}) \rightsquigarrow \langle\varphi_1, \ldots, \varphi_n\rangle$. A formula $A(\vec{x})$ is *modally definable* if it corresponds to some tuple of modal formulas. For $n = 1$ this yields the classical definition of local correspondence between a FO formula $A(x)$ with one free variable and a modal formula φ.

Let us write $F, \theta \models e{:}\,\varphi$ as a shortcut for $F, \theta, e \models \varphi$ and allow for disjunctions of expressions $e{:}\,\varphi$. Then we can rewrite the above equivalence as follows:

$$F \models A(e_1, \ldots, e_n) \iff \text{for every valuation } \theta \text{ we have } F, \theta \models e_1{:}\,\varphi_1 \vee \cdots \vee e_n{:}\,\varphi_n.$$

Or even shorter, using the notion of validity (see also Definition 2.4 below):

$$F \models A(e_1, \ldots, e_n) \iff F \models e_1{:}\,\varphi_1 \vee \cdots \vee e_n{:}\,\varphi_n.$$

[1] We use R_ℓ for a relation in a frame and R_ℓ for the corresponding predicate symbol.