Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Logic 4 (2006) 396–413
JOURNAL OF APPLIED LOGIC
www.elsevier.com/locate/jal

An example of formalizing recent mathematical results in MIZAR ☆

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Available online 10 November 2005

Abstract

This paper describes an example of the successful formalization of quite advanced and new mathematics using the MIZAR system. It shows that although much effort is required to formalize nontrivial facts in a formal computer deduction system, still it is possible to obtain the level of full logical correctness of all inference steps. We also discuss some problems encountered during the formalization, and try to point out some of the features of the MIZAR system responsible for that. The formalization described in this paper allows also for contrasting the linguistic capability of the MIZAR language and some of the phrases commonly used in "informal" mathematical papers that the MIZAR system lacks, and consequently presents the methods of how to cope with it during the formalization. Yet, apart from the problems, this paper shows some definite benefits from using a formal computer system in the work of a mathematician.

Keywords: Formalizing mathematics; MIZAR

1. Introduction

One of the biggest problems that worry the developers of automated deduction systems is that their systems are not sufficiently recognized and exploited by working mathematicians. Unlike the computer algebra systems, the use of which has indeed become ubiquitous in the work of mathematicians these days, deduction systems are still seldom

☆ This work is partially supported by the EU FP6 project Types for Proofs and Programs (TYPES).
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1570-8683/$ – see front matter
doi:10.1016/j.jal.2005.10.003

used. They are mostly used to formalize proofs of well-established and widely known classical theorems, the Fundamental Theorem of Algebra formalized in the systems Coq (see [3]) and MIZAR (see [6]) may serve as a perfect example here. Such work, however, is not always considered to be really challenging from the viewpoint of mathematicians who are concerned with obtaining their own new results. Therefore it has been recognized as a big challenge for the deduction systems community to prove that some of the state-of-the-art systems are developed enough to cope with formalizing recent mathematics.

This paper tries to support the point made above by showing an example of the successful formalization of quite advanced and new mathematics. It allows to observe that although much effort is required to formalize nontrivial facts in a formal computer deduction system, still it is possible to obtain the level of full logical correctness of all inference steps. The work is mostly based on the original results published recently in a renowned mathematical journal (see [14]) by the author of this article and K. Prażmowski. The formalization effort essential for this paper has been carried out using the MIZAR language as the formal background. This allowed to make use of the author's experience since 1996 first as a MIZAR user and then a member of the developers' team, as well as to draw some conclusions from his participation in several big formalization projects ongoing in the MIZAR society. The work is reflected in the articles [9–12]. All the formal proofs of theorems leading to the final result have been checked for full logical correctness by the MIZAR verifier and submitted to the MIZAR Mathematical Library (MML).

2. The theory formalized

The mathematical theory formalized in this experiment has been chosen to best serve its aims. It is abstract and general, so its description in a formal language was feasible. Moreover, as most geometrical theories it is both precise and intuitive. On the other hand, it is quite advanced and rather specialized. Namely, we formalized the concept of partial linear spaces and the Segre product of them. As a key result, we provided the codification of the theorem characterizing automorphisms (collineations) of the Segre product of strongly connected partial linear spaces based on the results published in [14]. Most known partial linear spaces (in particular, spaces of pencils being a generalization of the projective space) are strongly connected, so the result is quite general.

First, we introduced the axiomatic basis of partial linear spaces. The complete theory was built from scratch on top of MML, based only on the notion of a structure equipped with a fixed set and a family of its subsets, already present in MML. Then we constructed the notion of the Segre product. Originally, the Segre product, closely connected with the Segre variety, was defined for projective spaces (see [20]). In such a framework, the product can be considered as a geometrical counterpart of the tensor product operation defined for vector spaces. But it is seen that the Segre product can be defined for arbitrary (even partial) linear spaces and then it becomes a product in a suitable category. The approach used in this work is even more general. We defined the Segre product for a family of partial linear spaces indexed by an arbitrary non-empty set and proved that it satisfies the axioms of a partial linear space.