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Fresh Logic: proof-theory and semantics for FM and nominal techniques

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Abstract

In this paper we introduce *Fresh Logic*, a natural deduction style first-order logic extended with term-formers and quantifiers derived from the FM-sets model of names and binding in abstract syntax. Fresh Logic can be classical or intuitionistic depending on whether we include a law of excluded middle; we present a proof-normalisation procedure for the intuitionistic case and a semantics based on Kripke models in FM-sets for which it is sound and complete.

Keywords: Nominal logic; First-order logic; Semantics of intuitionistic logic; Kripke semantics; Freshness; Nominal sets; Fraenkel–Mostowski techniques

1. Introduction

In this paper we introduce *Fresh Logic*, a natural deduction style first-order logic extended with term-formers and quantifiers derived from the FM-sets model of names and binding in abstract syntax [9,11,12], developed by the author with A.M. Pitts. We show a proof-normalisation algorithm exists, and demonstrate soundness and completeness with respect to a model in FM-sets.

'FM' stands for 'Fraenkel–Mostowski', after two authors of papers on set theories very similar to what we call FM-sets (they were interested in proving independence of axioms of set theory [2], in particular the axiom of choice). Recently 'Nominal' has replaced 'FM' as an umbrella term for the logics and other systems derived from the original FM sets model, probably because it has fewer syllables and no umlauts.

Since this paper is a rock-hard proof-theoretic/semantic macho sets-fest, we keep the umlauts too.

FM-sets models names by a dedicated countably infinite set of names $a, b, c \in \mathbb{A}$, we give them a technical nomenclature *atoms*. This appears in Fresh Logic as a dedicated *type of atoms* $\mathbb{A}$ and a countably infinite set of atoms constant symbols $a, b, c, \ldots$ (they are not necessarily quite constant symbols, but we shall dis cuss that in due course).

FM-sets have a swapping action allowing us to rename (by swapping) atoms in sets. Fresh Logic has a corresponding term-former whose behaviour in the deduction system is to swap atoms constant symbols in terms. It is a fact that

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semantic swapping in FM-sets has excellent logical properties [12,26]—Fresh Logic is designed to take advantage of this in its proof-theory.

Finally, FM-sets give rise to a derived quantifier N meaning 'for all but a finite set of names'. Fresh Logic adds a quantifier N to the usual $\forall$ and $\exists$ of first-order logic. We shall see how swapping is vital to proof-normalisation for N, lending proof-theoretic support to the FM-sets model.

Fresh Logic can be classical or intuitionistic depending on whether we include a law of excluded middle; we present a proof-normalisation procedure for the intuitionistic case and a semantics based on Kripke models in FM-sets for which it is sound and complete. The core of Fresh Logic is first-order logic, unchanged.

The style of FM techniques has been "to be within ϵ of standard practice". The most developed FM system is FreshML [6]. FreshML programs look just like the informal specifications we would normally write; issues of α-equivalence and renaming are smoothly delegated to the compiler. In the same spirit, we want

> a logic whose judgements look like normal First-Order Logic, whose language is augmented with the FM N-quantifier, and with a good proof theory and semantics.

'Good' proof theory here means proof normalisation, Theorem 8.9, and also perhaps that the proof be simple and follow standard lines. Similarly for semantics. Soundness and completeness Theorem 7.7 and Theorem 9.12 follow standard lines, up to a point.

1.1. Outline of the paper

Section 2 We introduce a selection of relevant basic notions of FM-sets semantics (developed in detail in [12,28]); semantic atoms $\mathbb{A}$, sets with a swapping action, support as a syntax-free notion of 'free names of', the finite support property, the N quantifier, the difference set of two permutations of $\mathbb{A}$, equivariant functions, and freshness #.
Section 3 We define the terms, predicates, and contexts of Fresh Logic.
Section 4 Fresh Logic is intuitionistic. Its semantics consists of Kripke models of so-called 'possible worlds', which we call 'frames'; we define their structure.
Section 5 We define judgements, and inductively define valid judgements. This includes intro- and elim-rules for the N quantifier.
Section 6 We give some example derivations.
Section 7 We develop a notion of validity for judgements with respect to a Kripke model semantics whose possible worlds are frames and whose accessibility is a notion of frame map.
Section 8 We give a proof-normalisation algorithm for Fresh Logic.
Section 9 We discuss how to build a standard model of a Fresh Logic theory out its syntax and use this model to prove completeness. We discover that completeness demands we add an extra axiom (Small), which we analyse.
Section 10 We discuss alternatives to the many design decisions involved in making Fresh Logic.

Aside from a conclusions and related work section, there are also two appendices: Appendix A includes some definitions deferred from the text, Appendix B is a sequent-style presentation of Fresh Logic. We have not proved cut-elimination for the sequent presentation in full detail, but we have investigated it and expect it to work smoothly.

2. Semantics

Definition 2.1 *(The basics).* Fix a countably infinite *set of atoms* $a, b, c, \ldots \in \mathbb{A}$. For $a, b \in \mathbb{A}$ a *swapping* $(a\,b)$ is a function from $\mathbb{A}$ to $\mathbb{A}$ defined by

$$\begin{aligned}(b\,a)\cdot a &\stackrel{\text{def}}{=} b\\ (b\,a)\cdot b &\stackrel{\text{def}}{=} a \qquad\qquad (1)\\ (b\,a)\cdot c &\stackrel{\text{def}}{=} c \quad c \not\equiv a, b.\end{aligned}$$

Write $P_{\mathbb{A}}$ for *the set of finite permutations of* $\mathbb{A}$; the group generated by the $(a\,b)$ with functional composition $\circ$ as the group action. Write **Id** for the identity function on $\mathbb{A}$, which is of course the unit of the group $(P_{\mathbb{A}}, \circ)$.