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Journal of Applied Logic 4 (2006) 1–25

JOURNAL OF APPLIED LOGIC

www.elsevier.com/locate/jal

Essential unifiers

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Available online 29 January 2005

Abstract

A substitution δ is less general than a substitution σ if there exists λ such that $\delta = \sigma \cdot \lambda$, which induces a notion of generality in the algebra of substitutions. We propose to look at this well known concept of generality again, and to impose a new quasi ordering on substitutions as a natural result of a stronger notion of the composition of substitutions. This new generality ordering has important consequences for the theory of E-unification (unification in equational theories) and changes the basic notion of the most general unifiers, now called *essential unifiers*, as well as the unification hierarchy. In particular we show that for idempotent Semigroups (associativity and idempotency), also known as Bands, the set of essential unifiers always exists and is finite.

Keywords: E-unification; Equational theory; Substitutions; Terms; Unification theory; Part ordering; Universal algebra; Varieties

1. Introduction

Unification is an established concept in automated theorem proving, universal algebra, and semantics of non-imperative programming languages. Surveys of unification can be found in [2,3,14]. A survey of the related topic of rewriting systems is presented in [5] and for instance in [8].

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1570-8683/$ – see front matter
doi:10.1016/j.jal.2004.12.001

Unification solves equational problems and for practical applications it is often crucial to have a finite or at least minimal representation of all the solutions, i.e., a minimal complete set of unifiers from which all other solutions (unifiers) can be derived.

For equational problems in a free algebra of terms (also known as syntactic unification), there exists a unique unifier from which all others can be derived by instantiation, [10]. This is called the *most general unifier*. For equational algebras however the situation is completely different: a minimal complete set of unifiers does not always exist which was conjectured by Gordon Plotkin in his seminal paper in 1972, [9]. Since then unification problems and the underlying equational theories have been classified with respect to the cardinality of the minimal complete set of unifiers. These considerations enabled the development of general approaches and algorithms which apply to a whole class of theories, a topic of universal unification, see, e.g., the chapter "General Theory" in [14].

The contribution of this paper is a refinement of the important generating concept, now based on *essential unifiers*, which will narrow the set of most general unifiers.

We introduce an enhanced concept of a minimal generating set, which is based on a new notion of instantiation, but nevertheless complies with the notion of the composition of substitutions.

2. Basic notions and notations

To be self-contained and for the readers convenience, a set of common notions and definitions are presented in the following subsection.

2.1. Common definitions

An alphabet $\mathcal{F} = (F_n)_{n\in\mathcal{N}}$ provides a (finite) vocabulary. The function symbols in the sub-alphabet F_i, $i \in \mathcal{N}$, have the *arity* i. Function symbols with arity 0 are called *constants*. The set X is a denumerable set of variable symbols, called *variables*, usually denoted as x, y, z, etc. $\mathcal{F}$ and X constitute the *signature* of a term algebra.

The set of (first-order) terms $\mathcal{T}_{F,X}$ over a signature $\mathcal{F}$ generated by the variables X, is the smallest set containing the variables $x \in X$, and the terms $f(t_1, \ldots, t_n)$, whenever $f \in F_n$ is a function symbol of arity n and $t_1, \ldots, t_n \in \mathcal{T}_{F,X}$ are (recursively) terms. The set of terms is a (*free*) *term algebra*.

The set of variable-free terms are called *ground terms*. Terms that contain variables are said to be *open*. The set of *variables* occurring in a term t is denoted by $\mathbf{Var}(t)$ and the set of *symbols* of $\mathcal{F}$ occurring in t is denoted by $\mathbf{Sym}(t)$. This notion is extended to sets of variables, sets of terms, equations, and sets of equations, as usual.

For a term t the set of *sub-terms* $\mathbf{Sub}(t)$ contains $t \in \mathrm{Sub}(t)$ itself and is closed recursively by containing $t_1, \ldots, t_n \in \mathrm{Sub}(t)$, if $f(t_1, \ldots, t_n) \in \mathrm{Sub}(t)$. For a set of terms $T = \{t_1, t_2, \ldots, t_n\}$ the sub-terms are defined by $\mathrm{Sub}(T) = \mathrm{Sub}(t_1) \cup \cdots \cup \mathrm{Sub}(t_n)$.

A term t may be viewed also as a finite and labelled ordered tree, where the leaves are labelled with variables or constants, and the intermediate nodes of which are labelled with function symbols of positive arity.