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MONOTONICITY IN ORLICZ-LORENTZ SEQUENCE SPACES EQUIPPED WITH THE ORLICZ NORM*

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Abstract In Orlicz-Lorentz sequence space $\lambda_{\varphi,\omega}^{\circ}$ with the Orlicz norm, uniform monotonicity, points of upper local uniform monotonicity and lower local uniform monotonicity are characterized. Moreover, the monotonicity coefficient in $\lambda_{\varphi,\omega}^{\circ}$ are discussed.

Key words Orlicz-Lorentz sequence space; Orlicz norm; point of upper (lower) local uniform monotonicity; uniform monotonicity; monotone coefficient

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1 Introduction

A Banach lattice X with a lattice norm $\|\cdot\|$ is said to be strictly monotone (STM for short) [1] if for any $x \in X^+$(positive cone in X) and any $y \in X^+\backslash\{0\}$, we have $\|x+y\| > \|x\|$. A point $x \in S(X^+) := S(X)\cap X^+$ is said to be upper monotone [2] if, for any $y \in X^+\backslash\{0\}$, $\|x+y\| > 1$. A point $x \in S(X^+)$ is said to be lower monotone [2] if, for any $y \in X^+\backslash\{0\}$ and $y \leq x$, $\|x - y\| < 1$. An equivalent condition for X being strictly monotone [1] is that any point $x \in S(X^+)$ is lower monotone. But lower monotone points and upper monotone points are different, see [2]. X is called upper locally uniformly monotone (ULUM) [3] if for any $\varepsilon > 0$ and $x \in S(X^+)$, there exists $\delta(x,\varepsilon) > 0$ such that $y \in X^+$ and $\|y\| \geq \varepsilon$ imply $\|x+y\| \geq 1+\delta(x,\varepsilon)$. If for any $\varepsilon > 0$ and $x \in S(X^+)$, there is $\delta(x,\varepsilon) > 0$ such that $\|x-y\| \leq 1-\delta(x,\varepsilon)$ whenever $y \in X^+$, $\|y\| \geq \varepsilon$ and $y \leq x$, then X is said to be lower locally uniformly monotone (LLUM) [3]. We can analogously define points of lower local uniform monotonicity and points of upper local uniform monotonicity. We say that X is uniformly monotone (UM) [4] if for any $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) \in (0,1)$ such that $\|x+y\| > 1+\delta(\varepsilon)$ whenever $x \in S(X^+), y \in X^+$ and $\|y\| \geq \varepsilon$. For $\varepsilon \in [0,1]$, define $\eta_X(\varepsilon) = \inf\{\|x+y\|-1 :\ x,y \in X^+, \|x\| = 1, \|y\| \geq \varepsilon\}$. We call $m(X) = \sup\{\varepsilon \in [0,1] : \eta_X(\varepsilon) = 0\}$ the monotone coefficient [5] of X.

It is well known that some rotundity properties of Banach spaces were widely applied in ergodic theory, fixed point theory, probability theory and approximation theory, and in many cases these rotundity properties can be replaced by respective monotonicity properties when we

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restrict ourselves to a Banach space being a Banach lattice [3]. Roughly speaking, monotonicity properties played in Banach lattices similar role as rotundity properties in Banach spaces, and so for monotonicity points and rotundity points. Therefore in recent years monotonicity properties and monotonicity points were widely investigated in Musielak-Orlicz, Orlicz-Lorentz, Orlicz-Sobolev, Calderón-Lozanovskiǐ spaces [2, 3, 7, 8, 19]. In addition, some geometric properties concerning with the dual spaces of Orlicz-Lorentz spaces were researched by many mathematicians, where the Orlicz norm play a important role. In this paper we mainly give the criteria for Orlicz-Lorentz sequence spaces $\lambda^{\circ}_{\varphi,\omega}$ with the Orlicz norm being UM, a point in the space being upper locally uniformly monotone and lower locally uniformly monotone. At last we get the monotone coefficients of Orlicz-Lorentz sequence spaces with the Luxemburg norm and the Orlicz norm.

Let $\mathbb{R}, \mathbb{R}_+, \mathbb{N}$ be the set of all reals, nonnegative reals and natural numbers, respectively. A function $\varphi : \mathbb{R} \to \mathbb{R}_+$ is called an Orlicz function if φ is convex, even, $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, $\lim\limits_{u\to 0} \frac{\varphi(u)}{u} = 0$, and $\lim\limits_{u\to\infty} \frac{\varphi(u)}{u} = \infty$. Its Young's conjugate function ψ is defined by $\psi(v) = \sup\limits_{u>0}\{|uv| - \varphi(u)\}$, and ψ is also an Orlicz function. We say that $\omega = (\omega(i))$ is a weight sequence if ω is a non-increasing sequence of positive real numbers such that $\sum\limits_{i=1}^{\infty} \omega(i) = \infty$. The weight sequence ω is called regular if there exists $K > 1$ such that $S(2n) \geq KS(n)$ for any $n \in \mathbb{N}$, where $S(n) = \sum\limits_{i=1}^{n} \omega(i)$. For $x = (x(i))$ define the distribution function

$$d_x(\theta) = \mu\{i \in \mathbb{N} : |x(i)| > \theta\}, \qquad \theta \geq 0,$$

and the non-increasing rearrangement of x,

$$x^*(i) = \inf\{\theta > 0 : d_x(\theta) < i\}, \quad i \in \mathbb{N}.$$

For $x = (x(i))$, we call $\rho_{\varphi,\lambda}(x) = \sum\limits_{i=1}^{\infty} \varphi(x^*(i))\omega(i)$ the modular of x. The Orlicz-Lorentz sequence space $\lambda^{\circ}_{\varphi,\omega}$ generated by φ and ω is a Banach space

$$\lambda^{\circ}_{\varphi,\omega} = \{x = (x(i)) : \rho_{\varphi,\omega}(\lambda u) < \infty \text{ for some } \lambda > 0\}$$

endowed with the Orlicz norm [9]

$$\|x\|^{\circ}_{\varphi,\omega} = \sup_{\rho_{\psi,\omega}(y)\leq 1} \sum_{i=1}^{\infty} x^*(i)y^*(i)\omega(i),$$

or the Banach space $\lambda_{\varphi,\omega}$ equipped with the Luxemburg norm [10]

$$\|x\|_{\varphi,\omega} = \inf\Big\{\lambda > 0 : \rho_{\psi,\omega}(\frac{x}{\lambda}) \leq 1\Big\}.$$

Similarly as in the Orlicz space theory [11], denote

$$\theta_{\varphi,\omega}(x) := \inf\Big\{\lambda > 0 : \rho_{\varphi,\omega}(\frac{x}{\lambda}) < \infty\Big\}.$$

Recall that φ satisfies δ_2-condition if there exist $k > 0$ and $u_0 > 0$ such that $\varphi(2u) \leq k\varphi(u)$ for all $0 < u \leq u_0$. In this paper, denote $S_x := \{i \in \mathbb{N} : x(i) \neq 0\}$ for $x = (x(1), x(2), \cdots)$, and by μA the counting measure of A for $A \subset \mathbb{N}$. For more properties about non-increasing rearrangement and Orlicz-Lorentz spaces, we refer to [7, 10, 12–22].