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LARGE TIME BEHAVIOR OF A THIRD GRADE FLUID SYSTEM*

Xiaojuan CHAI (柴晓娟) *Zhengzheng CHEN* (陈正争) *Weisheng NIU* (钮维生)†
School of Mathematical Sciences, Anhui University, Hefei 230601, China
E-mail: *chaixj@ahu.edu.cn; chenzz@ahu.edu.cn; niuwsh@ahu.edu.cn*

Abstract We consider the large time behavior of a non-autonomous third grade fluid system, which could be viewed as a perturbation of the classical Navier-Stokes system. Under proper assumptions, we firstly prove that the family of processes generated by the problem admits a uniform attractor in the natural phase space. Then we prove the upper-semicontinuity of the uniform attractor when the perturbation tends to zero.

Key words third grade fluid equations; uniform attractor; upper-semicontinuity

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1 Introduction

Let $\Omega=[0,L]^d\subseteq\mathbb{R}^d, d=2$ or $3, L>0$. Let us consider the following system

$$\partial_t u-\nu\Delta u+(u\cdot\nabla)u-\alpha\mathrm{div}(A^2(u))-\beta\mathrm{div}(|A(u)|^2A(u))+\nabla\pi=f(x,t), \tag{1.1}$$

$$\mathrm{div}u=0,\ u|_{t=\tau}=u_\tau,\tau\in\mathbb{R}, \tag{1.2}$$

$$u(x+Le_j)=u(x), j=1,\cdots,d, \tag{1.3}$$

where $\nu>0$ is the constant kinematic viscosity, α,β are material constants and $\beta>0$. The unknown functions u,π are, respectively, the fluid velocity and the pressure. u_0,f are given functions representing, respectively, the initial fluid velocity and the forcing term. Here, A is the tensor defined as $A(u)=\nabla u+(\nabla u)^T$ with $(\nabla u)^T$ being the transposition of the Jacobian matrix ∇u, and $|A(u)|^2$ denotes $tr(A^2(u))$.

Equation (1.1) is a special case of the following third grade fluid equation

$$\partial_t v-\nu\Delta u+(u\cdot\nabla)v+\sum_j v_j\nabla u_j-(\alpha_1+\alpha_2)\mathrm{div}(A^2(u))-\beta\mathrm{div}(|A(u)|^2A(u))+\nabla\pi=f,$$

$$v=u-\alpha_1\Delta u.$$

Setting $\alpha_1=0$ in the equation above (see [19] for the physical meaning of this assumption), we obtain equation (1.1) immediately. The third grade fluid is an important case of the non-Newtonian fluids of grade n introduced by Rivlin and Ericksen [28], for which the stress tensor

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†Corresponding author: Weisheng NIU.

is a polynomial of degree n in the first n Rivlin-Ericksen tensors defined recursively by

$$A_1(u) = A(u) = \nabla u + (\nabla u)^T,$$
$$A_{k+1}(u) = (\partial_t + u \cdot \nabla)A_k(u) + (u \cdot \nabla)A_k(u) + (\nabla u)^T A_k(u) + A_k(u)\nabla u.$$

In the past years, the third grade fluid equation was studied largely. A detailed thermodynamic analysis for the model was given by Fosdick and Rajagopal in [12]. In [1, 30], some local existence and uniqueness results for the Cauchy problem with initial data of arbitrary size, or the global existence and uniqueness results for the Cauchy problem with small initial data were obtained in the whole space $\mathbb{R}^d, d = 2, 3$. These results were then improved by Busuioc and Iftimie in [2], and by Paicu in [25]. In [16], Hamza and Paicu studied the existence and uniqueness result of the Cauchy problem corresponding to equation (1.1) in $\mathbb{R}^3$. The authors proved that the Cauchy problem is globally well posed in the energy space $\mathbb{L}^2(\mathbb{R}^3)$. Then, in [37] Zhao and his coauthors investigated the time decay results of the weak solution to the Cauchy problem considered in [16]. They provided the upper and lower bounds for the time decay rate in the energy space $\mathbb{L}^2(\mathbb{R}^3)$. More recently, in [7], we investigated the large time behavior of an autonomous conducting third grade fluid system in the presence of a magnetic field. We proved the existence of a global attractor and an exponential attractor in proper product spaces.

In this paper, we consider the large time behavior of solutions to the non-autonomous third grade fluid system (1.1)–(1.3). The non-autonomous infinite dynamical systems were largely studied in the past decades. Paralleling to the notion of global attractors for the autonomous dynamical systems, the notions of uniform attractors and pullback attractors were proposed and well developed for non-autonomous dynamical systems . The existence and some properties (such as the structure, the dimension and the stability property, ect.) of uniform attractors or pullback attractors were investigated for various non-autonomous systems, such as the incompressible Navier-Stokes systems, reaction diffusion systems, damped wave equations and so on, see [8–10, 15, 18, 27, 39] and the large amount of references therein. In the present paper, we will mainly consider the existence of uniform attractors for the non-autonomous third grade fluid system (1.1)–(1.3).

Throughout the paper, we make the assumption that the initial data and forcing term are zero-spatial-mean functions ($\int_\Omega \varphi \mathrm{d}x = 0$ for $\varphi = u_0, f$), so that the solutions have zero spatial mean, which allows us using the Poincaré inequality. As we are interested in the non-autonomous case, thus we assume that the forcing term f is time dependent. With proper assumptions on the material constants α, β and the forcing term f, we first provide the existence and uniqueness result for the system with $\mathbb{L}^2$ initial data. Then, we prove the existence of a uniform attractor in $\mathbb{L}^2$.

Note that when the material constants α, β in (1.1)–(1.3) tend to zero, the system converges formally to the Navier-Stokes system

$$\begin{cases} \partial_t u - \nu\Delta u + (u \cdot \nabla)u + \nabla\pi = f(x, t), \\ \mathrm{div} u = 0, \quad u|_{t=\tau} = u_\tau, \tau \in \mathbb{R}, \\ u(x + Le_j) = u(x), j = 1, \cdots, d. \end{cases} \tag{1.4}$$

One may expect that the solutions of system (1.1)–(1.3) converge to solutions of the Navier-Stokes system, and so does the corresponding attractor. Thus, the second aim of this paper is