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RIGIDITY OF COMPACT SURFACES IN HOMOGENEOUS 3-MANIFOLDS WITH CONSTANT MEAN CURVATURE*

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Abstract In this paper, we establish a rigidity theorem for compact constant mean curvature surfaces of the Berger sphere in terms of the surfaces' geometric invariants. This extends the previous similar result on compact minimal surfaces of the Berger sphere.

Key words homogeneous 3-manifolds; Berger sphere; constant mean curvature surface; Hopf torus; Clifford torus

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1 Introduction

This paper is a continuation of Hu-Lyu-Wang's previous work [1]. According to the standard notation, we denote by $\mathbb{E}(\kappa,\tau)$ the homogeneous 3-manifolds whose isometry group is of dimension 4, where κ and τ are constant and $\kappa\neq 4\tau^2$. In [1] the authors studied surfaces of $\mathbb{E}(\kappa,\tau)$ and, as main results, rigidity theorems in terms of the second fundamental form are established for compact minimal surfaces of the Berger sphere $\mathbb{S}^3_b(\kappa,\tau)$ $(\kappa\neq 4\tau^2)$. In this paper, by checking the proof of [1] in further detail, we succeed in extending the rigidity theorem therein to all compact surfaces of $\mathbb{S}^3_b(\kappa,\tau)$ $(\kappa\neq 4\tau^2)$ with constant mean curvature.

We noticed that, in the last years the study of constant mean curvature surfaces of the homogeneous Riemannian 3-manifolds is a topic of increasing interest, see [2–11] and references therein.

Recall that for $\mathbb{E}(\kappa,\tau)$, equipped with a Riemannian metric g that also denoted by $\langle\cdot,\cdot\rangle$, there exists a Riemannian submersion $\Pi:\mathbb{E}(\kappa,\tau)\to\mathbb{M}^2(\kappa)$, where $\mathbb{M}^2(\kappa)$ is a 2-dimensional simply connected space form of constant curvature κ, with totally geodesic fibers and there exists a unit Killing vector field ξ on $\mathbb{E}(\kappa,\tau)$ which is vertical with respect to Π. The bundle curvature is the number τ such that $\overline{\nabla}_X\xi=\tau X\times\xi$ for any vector field X on $\mathbb{E}(\kappa,\tau)$, where $\times$

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denotes the vector product and $\overline{\nabla}$ denotes the Riemannian connection of $\mathbb{E}(\kappa,\tau)$, respectively. When $\tau=0$ (and then $\kappa\neq 0$), we get a product manifold $\mathbb{M}^2(\kappa)\times\mathbb{R}$, the vertical vector ξ is tangent to the factor $\mathbb{R}$. This case was treated extensively, we refer to [12], among many others. The manifolds with $\tau\neq 0$ are of three types based on the value of κ: the Berger spheres for $\kappa>0$ that is commonly denoted by $\mathbb{S}^3_b(\kappa,\tau)$; the Heisenberg group Nil_3 for $\kappa=0$; and the universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the Lie group $\mathrm{SL}(2,\mathbb{R})$ (endowed with a 2-parameter family of homogeneous metrics) for $\kappa<0$.

To state our results, let $\Phi:\Sigma\to\mathbb{E}(\kappa,\tau)$ be an immersion of a surface Σ, and denote by C, H and S the angle function, the mean curvature and the squared norm of the second fundamental form of Φ, respectively (see Section 2 for the definition of C, H and S). Then the main result of [1] can be stated as

Theorem 1.1 (see [1]) Let $\Phi:\Sigma\to\mathbb{E}(\kappa,\tau)$ be a minimal immersion of a compact surface Σ. Then it holds the Simons' type integral inequality

$$\int_\Sigma\Big\{S^2-[2\tau^2+(\kappa-4\tau^2)(5C^2-1)]S+2(\kappa-4\tau^2)\tau^2(3C^2-1)\Big\}\mathrm{d}\sigma\geq 0, \tag{1.1}$$

where the equality holds if and only if Φ is of parallel second fundamental form.

In particular, if $\Phi:\Sigma\to\mathbb{S}^3_b(\kappa,\tau)$ $(\kappa\neq 4\tau^2)$ is a minimal immersion of a compact surface Σ, then equality holds in (1.1) if and only if $\Phi:\Sigma\to\mathbb{S}^3_b(\kappa,\tau)$ is the Clifford torus, the latter case occurs only when $C\equiv 0$ and $S\equiv 2\tau^2$.

To prove Theorem 1.1, due to that $\mathbb{E}(\kappa,\tau)$ is of no constant sectional curvature, instead of computing the Laplacian of the squared norm of the second fundamental form ΔS alone as one usually did when studying minimal surfaces of the unit sphere $\mathbb{S}^3(1)$, a computation of the combination $\Delta S-(\kappa-4\tau^2)[\Delta|T|^2-2\mathrm{div}(\nabla_T T)]$ was carried out, here T denotes the tangential component of the Killing field ξ onto the surface.

Conceptually, one expects that the rigidity phenomena of Theorem 1.1 can be extended to all surfaces of $\mathbb{E}(\kappa,\tau)$ with constant mean curvature. We find, however, if we deal again with the combination $\Delta S-(\kappa-4\tau^2)[\Delta|T|^2-2\mathrm{div}(\nabla_T T)]$ in the situation of non-minimal constant mean curvature surfaces, it turns out impossible. Eventually, we find that the right way is to compute $\Delta S-(\kappa-4\tau^2)\Delta|T|^2$. Accordingly, in this paper we can extend Theorem 1.1 to achieve the following main conclusion.

Theorem 1.2 Let $\Phi:\Sigma\to\mathbb{E}(\kappa,\tau)$ be a constant mean curvature immersion of a compact surface Σ. Then it holds the Simons' type integral inequality

$$\begin{aligned}\int_\Sigma\Big\{&(S-2H^2)^2-\big[2(\tau^2+H^2)+\frac{1}{2}(\kappa-4\tau^2)(7C^2-1)\big](S-2H^2)\\&+(\kappa-4\tau^2)(\tau^2+H^2)(3C^2-1)+(\kappa-4\tau^2)^2C^2(1-C^2)\Big\}\mathrm{d}\sigma\geq 0,\end{aligned} \tag{1.2}$$

where the equality holds if and only if Φ is of parallel second fundamental form.

In particular, if $\Phi:\Sigma\to\mathbb{S}^3_b(\kappa,\tau)$ $(\kappa\neq 4\tau^2)$ is a constant mean curvature immersion of a compact surface Σ, then equality holds in (1.2) if and only if $\Phi:\Sigma\to\mathbb{S}^3_b(\kappa,\tau)$ is the Clifford torus, the latter case occurs only when $C\equiv 0$ and $S\equiv 2\tau^2+4H^2$.

As a counterpart of Theorem 1.1, from Theorem 1.2 we immediately have the following.

Corollary 1.3 Let $\Phi:\Sigma\to\mathbb{E}(\kappa,\tau)$ be a minimal immersion of a compact surface Σ.