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A NONLOCAL HYBRID BOUNDARY VALUE PROBLEM OF CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we discuss the existence of solutions for a nonlocal hybrid boundary value problem of Caputo fractional integro-differential equations. Our main result is based on a hybrid fixed point theorem for a sum of three operators due to Dhage, and is well illustrated with the aid of an example.

Key words Caputo fractional derivative; integral; hybrid; fixed point theorem

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1 Introduction

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc. [1–3]. For some recent development on the topic, see [4–16] and the references therein.

Hybrid fractional differential equations were also studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers (see [17–21]).

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In this paper we study the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential equations given by

$$\begin{cases} {}^cD^{\alpha}\left[\dfrac{x(t)-\sum\limits_{i=1}^{m} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right]=g(t,x(t)), \quad t\in J:=[0,1], \\ x(0)=\mu(x), \quad x(1)=A, \end{cases} \tag{1.1}$$

where ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α, $1<\alpha\le 2$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi>0$, $\phi\in\{\beta_1,\beta_2,\cdots,\beta_m\}$, $f\in C(J\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g\in C(J\times\mathbb{R},\mathbb{R})$, $h_i\in C(J\times\mathbb{R},\mathbb{R})$, $0<\beta_i$, $i=1,2,\cdots,m$, $\mu: C([0,1],\mathbb{R})\to\mathbb{R}$ and $A\in\mathbb{R}$.

The rest of the paper is organized as follows. In Section 2, we recall some useful preliminaries. Section 3 contains the main result which is obtained by means of a hybrid fixed point theorem for three operators in a Banach algebra due to Dhage [22]. Also see the papers [23, 24]. An example is also discussed for illustration of the main result.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [1, 2] and present preliminary results needed in our proofs later.

Definition 2.1 For $(n-1)$-times absolutely continuous function $g:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t)=\frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)\mathrm{d}s, \quad n-1<q<n, n=[q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 The Riemann-Liouville fractional integral of order $p>0$ of a continuous function $f:(0,\infty)\to\mathbb{R}$ is defined by

$$I^p f(t)=\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}f(s)\mathrm{d}s,$$

provided the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3 The Caputo derivative of order q for a function $f:[0,\infty)\to\mathbb{R}$ can be written as

$${}^cD^q f(t)=D^q\left(f(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\right), \quad t>0, \quad n-1<q<n.$$

Remark 2.4 If $f(t)\in C^n[0,\infty)$, then

$$\begin{aligned} {}^cD^q f(t)&=\frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\mathrm{d}s \\ &=I^{n-q}f^{(n)}(t),\ t>0,\ n-1<q<n. \end{aligned}$$

Lemma 2.5 (see [1]) Let $x\in C^m[0,1]$ and $y\in AC[0,1]$. Then, for $q\in(m-1,m), m\in\mathbb{N}$ and $t\in[0,1]$,

(a) the general solution of the fractional differential equation ${}^cD^q x(t)=0$ is $x(t)=k_0+k_1t+k_2t^2+\cdots+k_{m-1}t^{m-1}$, where $k_i\in\mathbb{R},\ i=0,1,2,\cdots,m-1$;