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EXISTENCE OF WEAK SOLUTIONS FOR SOME SINGULAR PARABOLIC EQUATION*

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Abstract In this paper, we are concerned with a singular parabolic equation subject to Dirichlet boundary condition and initial condition. Under different assumptions on μ, ν and ψ, some existence results are obtained by applying parabolic regularization method and the sub-super solutions method.

Key words existence; singular equation; blowup

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1 Introduction

In this paper, we are concerned with existence of positive solutions for a singular parabolic equation

$$v_t - v'' - \frac{\mu}{r}v' + \nu\frac{|v'|^2}{v} = g(r,t), \quad v \geq 0, (r,t) \in (0,1) \times (0,T], \tag{1.1}$$

subject to the following boundary and initial conditions

$$v(0,t) = v(1,t) = 0, \quad t \in (0,T], \tag{1.2}$$

$$v(r,0) = \psi(r), \qquad r \in (0,1), \tag{1.3}$$

where μ, ν are nonnegative constants.

In [1], Zhou and Lei discussed the following heat equation with singular terms:

$$v_t - v'' + \nu v^m |v'|^2 = g(r,t), \quad (r,t) \in (a,b) \times (0,T],$$

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$$v(a,t)=v(b,t)=0, \qquad t\in(0,T], \tag{1.4}$$
$$v(r,0)=\psi(r), \qquad r\in(a,b),$$

here $\nu>0$, $m\in(-2,-1]$. They obtained existence of weak solutions and found that the problem may have multiple weak solutions for some initial data. Note that when $m=-1$, equation (1.4) turns to be equation (1.1) with $\mu=0$.

In [2], Xia etc. studied weak solutions of the following equation

$$v''+\frac{\mu}{r}v'-\nu\frac{|v'|^2}{v}+\lambda(r)=0, \tag{1.5}$$

subject to Dirichlet boundary condition

$$v(0)=v(1)=0,$$

where $\mu>0,\nu>\mu+1$ are constants, $c<\lambda(r)\in L^\infty(0,1)$ for some positive constant c. Note that equation (1.5) is the stationary case of equation (1.1).

Actually, equation (1.1) is closely related with other equations. For example, if $\mu=N-1$, suppose $g(|x|,t)$ is a radially symmetric function with respect to $x\in B_1\subset\mathbb{R}^N(N\geq 2)$, then equation (1.1) is related with following equation

$$\frac{\partial v}{\partial t}-\Delta v+\nu\frac{|\nabla v|^2}{v}=g(|x|,t), \quad (x,t)\in B_1\setminus\{0\}\times(0,T], \tag{1.6}$$

subject to the following conditions

$$v(|x|,t)=0, \qquad (x,t)\in\partial B_1\cup\{0\}\times(0,T],$$
$$v(|x|,0)=\psi(|x|), \quad x\in B_1,$$

where B_1 is the unit ball in $\mathbb{R}^N$, and we may redefine $r=|x|$. Equation (1.6) has many applications in physics, chemistry and biology, see [1–3] and the references please. Existence, uniqueness and asymptotic behavior of classic or weak solutions for equation (1.6) with general forms were discussed in recent years, we refer readers to [3–8] and their references.

In this paper we shall prove existence of weak solutions for problem (1.1)–(1.3) mainly by applying the classical parabolic regularization method and the sub-super solutions method. And in Section 2 we will list the main results and give the proof.

2 Main Results and Proof

First we define what we mean by a weak solution for problem (1.1)–(1.3). Throughout the paper, we denote $\Theta_T=(0,1)\times(0,T]$, $\overline{\Theta}_T=[0,1]\times[0,T]$, $C_c^\infty(A)$ is the space of infinitely differentiable functions with compact support in A. And we shall omit integral domain Θ_T in all integral equality or inequality. We also suppose $g(r,t)$ and ψ satisfy the following conditions:

(H1) $g(r,t)\in C(\overline{\Theta}_T), g(r,t)>0$ on $\overline{\Theta}_T$;

(H2) $\psi\in C[0,1]\cap H^1(0,1),\psi>0$ in $(0,1),\psi(0)=\psi(1)=0$.

Definition 2.1 Nonnegative function v is called a weak solution for problem (1.1)–(1.3), if

(a) $v>0$ a.e. in Θ_T;

(b) $v\in\mathbb{V}:=L^2(0,T;W_0^{1,2}(0,1))\cap L^\infty(\Theta_T)$, $v_t\in\mathbb{V}^*$ and $r^\mu|v'|^2\in L^1(\Theta_T)$;