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# A BIFURCATION PROBLEM ASSOCIATED TO AN ASYMPTOTICALLY LINEAR FUNCTION*

*Soumaya SÂANOUNI*
*Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, Campus University 2092 Tunis, Tunisia*
*E-mail*: *saanouni.soumaya@yahoo.com*

*Nihed TRABELSI*
*Higher Institut of Medicals Technologies of Tunis, University of Tunis El Manar, 9 Street Dr. Zouhair Essafi 1006 Tunis, Tunisia*
*E-mail*: *nihed.trabelsi78@gmail.com*

**Abstract** We study the existence of positive solutions to a two-order semilinear elliptic problem with Dirichlet boundary condition

$$(P_\lambda) \quad \begin{cases} -\operatorname{div}(c(x)\nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$; $n \geq 2$ is a smooth bounded domain; $f$ is a positive, increasing and convex source term and $c(x)$ is a smooth bounded positive function on $\overline{\Omega}$. We also prove the existence of critical value and claim the uniqueness of extremal solutions.

**Key words** extremal solution; regularity; bifurcation; stability

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## 1 Introduction and Statement of Main Results

The main interest of non-linear physics lies in its ability to explain the evolution of the problems: a phenomenon usually depends on a number of parameters, called control parameters that control the evolution of the system. By variation of the parameters and the result of non-linearities, the system may undergo transitions. In math, they are bifurcations.

In biology; population dynamics, the equations of reactions-diffusion were proposed by Turing (1952) for modeling morphogenesis phenomena. A model of interaction of species or chemicals is given by the differential equations

$$\frac{\partial u}{\partial t} = \operatorname{div}(c\nabla u) + \lambda f,$$

where $u$ represents the density, $\operatorname{div}(c\nabla u)$ represents the substance of diffusion through the system, and $f$ models the interaction of substances. Moreover the broadcast functions $c$ and the terms of reactions $f$ may depend on $(x, t)$ and the concentrations $u$ it as a nonlinearity way.

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In case where $c$ and $t$ are constants, various authors have studied the existence of weak solutions for the bifurcation problem

$$(E_\lambda)\quad \begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \ \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$.

Mironescu and Rădulescu proved in [13, 14] that there exists $0 < \lambda^* < \infty$, a critical value of the parameter $\lambda$, such that $(E_\lambda)$ has a minimal, positive, classical solution $u_\lambda$ for $0 < \lambda < \lambda^*$ and does not have a weak solution for $\lambda > \lambda^*$.

Our main interest here will be in the study of a bifurcation problem for $\lambda > 0$

$$(P_\lambda)\quad \begin{cases} -\mathrm{div}(c(x)\nabla u) = \lambda f(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \ \partial\Omega, \end{cases}$$

which is a generalization of $(E_\lambda)$. Where $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $c(x)$ is a smooth bounded positive function on $\overline{\Omega}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a positive, increasing and convex smooth function, such that

$$f(0) > 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{t} = a \in (0, \infty).$$

The value $a$ was be crucial in the study of $(E_{\lambda^*})$ and of the behavior of $u_\lambda$ when $\lambda$ tends to $\lambda^*$.

Throughout the paper, we denote by $\|\cdot\|_2$, the $L^2(\Omega)$-norm, whereas we denote by $\|\cdot\|$, the $H_0^1(\Omega)$-norm given by

$$\|u\|^2 = \int_\Omega |\nabla u|^2.$$

We say that $u \in H_0^1(\Omega)$ is a weak solution of the problem $(P_\lambda)$, if $f(u) \in L^1(\Omega)$ and

$$\int_\Omega c(x)\nabla u.\nabla\varphi = \lambda \int_\Omega f(u)\varphi, \quad \forall \varphi \in H_0^1(\Omega).$$

Such solutions are usually known as weak energy solutions. For short, we will refer to them simply as solutions wish is assuredly by the next lemma.

**Lemma 1.1** Since $f(t) \leq at + f(0)$, if $u \in L^1(\Omega)$ is a weak solution of $(P_\lambda)$, it is easily seen by a standard bootstrap argument that $u$ is always a classical solution.

In the rest of this paper, we denote by a solution of problem $(P_\lambda)$ any weak or classical solution.

**Definition 1.2** We say that a solution $u_\lambda$ of problem $(P_\lambda)$ is minimal if $u_\lambda \leq u$ in $\Omega$ for any solution $u$ of $(P_\lambda)$.

**Definition 1.3** We say that $u \in H_0^1(\Omega)$ is a supersolution of $(P_\lambda)$ if $f(u) \in L^1(\Omega)$ and

$$-\mathrm{div}(c(x)\nabla u) \geq \lambda f(u) \ \text{ in } \ \mathcal{D}'(\Omega) \quad \text{for any nonnegative test function.}$$

Reversing the inequality one defines the notion of subsolution.