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ON DOUBLY WARPED PRODUCT OF COMPLEX FINSLER MANIFOLDS*

Yong HE (何勇)[1,2] *Chunping ZHONG* (钟春平)[1]
1. School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
2. School of Mathematical Sciences, Xinjiang Normal University, Urumqi 830053, China
E-mail: *19020130154147@stu.xmu.edu.cn; zcp@xmu.edu.cn*

Abstract Let (M_1, F_1) and (M_2, F_2) be two strongly pseudoconvex complex Finsler manifolds. The doubly wraped product complex Finsler manifold $({}_{f_2}M_1 \times_{f_1} M_2, F)$ of (M_1, F_1) and (M_2, F_2) is the product manifold $M_1 \times M_2$ endowed with the warped product complex Finsler metric $F^2 = f_2^2 F_1^2 + f_1^2 F_2^2$, where f_1 and f_2 are positive smooth functions on M_1 and M_2, respectively. In this paper, the most often used complex Finsler connections, holomorphic curvature, Ricci scalar curvature, and real geodesics of the DWP-complex Finsler manifold are derived in terms of the corresponding objects of its components. Necessary and sufficient conditions for the DWP-complex Finsler manifold to be Kähler Finsler (resp., weakly Kähler Finsler, complex Berwald, weakly complex Berwald, complex Landsberg) manifold are obtained, respectively. It is proved that if (M_1, F_1) and (M_2, F_2) are projectively flat, then the DWP-complex Finsler manifold is projectively flat if and only if f_1 and f_2 are positive constants.

Key words doubly warped products; complex Finsler metric; holomorphic curvature; geodesic

2010 MR Subject Classification 53C60; 53C40

1 Introduction

Singly warped product or simply warped product of Riemannian manifolds was first defined by O'Neill and Bishop in [12] to construct Riemannian manifolds with negative sectional curvature, then in [22], O'Neill obtained the curvature formulae of warped products in terms of curvatures of its components. The recent studies showed that warped product is useful in theoretical physics, particulary in general relativity. For instance, Beem, Ehrlich and Powell [11] pointed out that many exact solutions to Einstein's field equation can be expressed in terms of Lorentzian warped products. Under the assumption that four-dimensional space-time to be a general warped product of two surfaces, Katanaev, Klösch and Kummer [16] explicitly constructed all global vacuum solutions to the four-dimensional Einstein equations. Doubly warped product of Riemannian manifolds was also of interesting and was studied by Unal [30].

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The notion of warped product was recently extended to Finsler spaces. In [6, 7], Asanov obtained some models of relativity theory which were described by the warped product of Finsler metrics. In [18], Kozma, Peter and Varga studied the warped product of real Finsler manifolds. They obtained the relationship between the Cartan connection of the warped product Finsler metric and the Cartan connections of its components. More recently, Hushmandi and Rezaii [8] studied the warped product Finsler spaces of Landsberg type, and then in [9], Hushmandi, Rezaii and Morteza obtained the curvature of warped product Finsler spaces and the Laplacian of the Sasaki-Finsler metrics. In [23], Peyghan and Tayebi obtained the relationship between the Riemannian curvature of the doubly warped product Finsler manifold and the curvatures of its components.

Let (M_1, F_1) and (M_2, F_2) be two complex Finsler manifolds. In [33], Wu and Zhong considered the product complex manifold $M = M_1 \times M_2$ endowed with a complex Finsler metric $F = f(K, H)$, where $f(K, H)$ is a function of $K = F_1^2$ and $H = F_2^2$. The possibility of F to be Kähler Finsler metric and complex Berwald metric were investigated. Recently in [35], Zhong showed that there are lots of strongly pseudoconvex (even strongly convex) unitary invariant complex Finsler metrics in domains in $\mathbb{C}^n$. In this paper, we consider the warped product of strongly pseudoconvex complex Finsler manifolds. Our purpose of doing this is to study the possibility of constructing some special strongly pseudoconvex complex Finsler metrics such as Kähler Finsler metrics, weakly Kähler Finsler metrics, complex Berwald metrics, complex Landsberg metrics and weakly complex Berwald metrics, among which to find possible way to obtain strongly pseudoconvex complex Finsler metrics which are of constant holomorphic curvatures. Note that it was prove in [33] that the Chern-Finsler nonlinear connection coefficients are independent of the choice of f, i.e., there is no difference between the case $F^2 = f(K, H)$ and $F^2 = F_1^2 + F_2^2$. In this paper, the warping product metric F on the product complex manifold $M = M_1 \times M_2$ is $F^2 = f_2^2 F_1^2 + f_1^2 F_2^2$, which generalizes [33] whenever f_1 and f_2 are not positive constants.

This paper is organized as follows. In Section 2, we recall some basic concepts and notions in complex Finsler geometry. In Section 3, we derive the most often used complex Finsler connections (the Chern-Finsler connection, the complex Rund connection, the complex Berwald connection, and the complex Hashiguchi connection, etc.) of the DWP-complex Finsler manifold in terms of the corresponding connections of its components, respectively. In Section 4, we derive the formulae of the holomorphic curvature and Ricci scalar curvature of the DWP-complex Finsler manifold in terms of the holomorphic curvatures and Ricci scalar curvatures of its components. In Section 5, we obtain the necessary and sufficient conditions for the DWP-complex Finsler manifold to be Kähler Finsler (resp. weakly Kähler Finsler, complex Berwald, weakly complex Berwald, complex Landsberg, complex locally Minkowski) manifold. In Section 6, we derive the real geodesic equations of the DWP-complex Finsler manifold in terms of the geodesic equations of its components, and prove that if the warping function f_1 (resp. f_2) is a positive constant, then (M_1, F_1) (resp. (M_2, F_2)) is a totally geodesic manifold of the DWP-complex Finsler manifold $({}_{f_2}M_1 \times_{f_1} M_2, F)$, and the projection of any geodesic of the DWP-complex Finsler manifold onto M_1 (resp. M_2) is a geodesic of (M_1, F_1) (resp. (M_2, F_2)). In Section 7, we prove that if (M_1, F_1) and (M_2, F_2) are locally projectively flat manifolds, then the DWP-complex Finsler manifold $({}_{f_2}M_1 \times_{f_1} M_2, F)$ is projectively flat if and only if f_1 and