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MULTIPLICITY RESULTS FOR A NONLINEAR ELLIPTIC PROBLEM INVOLVING THE FRACTIONAL LAPLACIAN*

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Abstract In this paper, we consider a class of superlinear elliptic problems involving fractional Laplacian $(-\Delta)^{s/2}u=\lambda f(u)$ in a bounded smooth domain with zero Dirichlet boundary condition. We use the method on harmonic extension to study the dependence of the number of sign-changing solutions on the parameter λ.

Key words fractional Laplacian; existence; asymptotic; Sobolev trace inequality

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1 Introduction

Problems of the type

$$\begin{cases} -\Delta u=\lambda f(u) \text{ in } & \Omega; \\ u=0 \qquad\qquad \text{on} & \Omega \end{cases} \tag{1.1}$$

for different kind of nonlinearities f, were the main subject of investigation in past decades. See for example the list [2, 4, 5, 10, 14, 16, 17]. Specially, in 1878, Rabinowitz [14] gave multiplicity results of (1.1) for any positive parameter λ as $n=1$. But he found that the number of solutions of (1.1) is independent on λ. Under some conditions on f, Costa and Wang [5] proved that the number of signed and sign-changing solutions is dependent on the parameter λ as $n\geq 1$.

Recently, fractional Laplacians attracted much interest in nonlinear analysis. Caffarelli et al.[7, 8] studied a free boundary problem. Since the work of Caffarelli and Silvestre [9], who

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introduced the s-harmonic extension to define the fractional Laplacian operator, several results of version of the classical elliptic problems were obtained, one can see [3, 6] and their references.

In this paper, we consider the nonlinear elliptic problem involving the fractional Laplacian power of the Dirichlet Laplacian

$$\begin{cases} (-\Delta)^{s/2}u = \lambda f(u), \text{ in } & \Omega; \\ u = 0, \qquad\qquad\quad \text{on} & \Omega, \end{cases} \tag{P$_\lambda$}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, λ is a positive parameter, $s \in (0, 2), (-\Delta)^{s/2}$ stands for the fractional Laplacian, and $f : \mathbb{R} \to \mathbb{R}$ satisfies:

$$f \in \mathcal{C}(\mathbb{R}) \quad \text{and} \quad \lim_{t\to 0} \frac{f(t)}{|t|^{p-2}t} = 1 \ \text{ for some } \ 2 < p < 2^{\#} = \frac{2n}{n-s}. \tag{F}$$

For the definition of fractional Laplacian operator we follow some idea of [3]. In particular, we define the eigenvalues ρ_k of $(-\Delta)^{s/2}$ as the power $s/2$ of the eigenvalues λ_k of $(-\Delta)$, i.e., $\rho_k = \lambda_k^{s/2}$ both with zero Dirichlet boundary data.

Let $N(\lambda)$ be the number of sign-changing solutions of (P_λ). Our main result is the following theorem.

Theorem 1.1 If f satisfies condition (F), then $\lim\limits_{\lambda\to+\infty} N(\lambda) = +\infty$.

2 Preliminaries

Denote the half cylinder with base on a bounded smooth domain Ω by

$$C = \Omega \times (0, \infty),$$

and its lateral boundary by

$$\partial_L C = \partial\Omega \times [0, \infty).$$

Let $\{\lambda_k, \varphi_k\}_{k=1}^{\infty}$ be the eigenvalues and corresponding eigenfunctions of the Laplacian operator $(-\Delta)$ in Ω with zero Dirichlet boundary values on $\partial\Omega$, such that $\|\varphi_k\|_{L^2(\Omega)} = 1$. Let

$$H_0^{s/2}(\Omega) = \left\{ u = \sum_{k=0}^{\infty} a_k\varphi_k \in L^2 : \|u\|_{H_0^{s/2}(\Omega)} = \left(\sum_{k=0}^{\infty} a_k^2 \lambda_k^{s/2} \right)^{1/2} < \infty \right\}. \tag{2.1}$$

Denote $H^{-s/2}(\Omega)$ the dual space of $H_0^{s/2}(\Omega)$. $(-\Delta)^{s/2}$ is given by

$$(-\Delta)^{s/2}u = \sum_{k=1}^{\infty} \alpha_k \lambda_k^{s/2} \varphi_k \tag{2.2}$$

for $u = \sum\limits_{k=1}^{\infty} \alpha_k\varphi_k \in L^2(\Omega)$.

Associated to problem (P$_\lambda$), the corresponding energy functional $I_1 : H_0^{s/2}(\Omega) \to \mathbb{R}$ is defined as follows:

$$I_1(u) = \frac{1}{2}\int_\Omega |(-\Delta)^{s/4}u|^2 \mathrm{d}x - \lambda \int_\Omega G(u)\mathrm{d}x, \quad \forall u \in H_0^{s/2}(\Omega), \tag{2.3}$$

where $G(u) = \int_0^u f(t)\mathrm{d}t$.