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GLOBAL WELL-POSEDNESS IN ENERGY SPACE OF SMALL AMPLITUDE SOLUTIONS FOR KLEIN-GORDON-ZAKHAROV EQUATION IN THREE SPACE DIMENSION*

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Dedicated to Professor Boling Guo on the occasion of his 80th birthday

Abstract The Cauchy problem of the Klein-Gordon-Zakharov equation in three dimensional space

$$\begin{cases} u_{tt}-\Delta u+u=-nu, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}_+,\\ n_{tt}-\Delta n=\Delta|u|^2, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}_+,\\ u(x,0)=u_0(x), \quad \partial_t u(x,0)=u_1(x), \quad n(x,0)=n_0(x), \quad \partial_t n(x,0)=n_1(x), \end{cases} \tag{0.1}$$

is considered. It is shown that it is globally well-posed in energy space $H^1\times L^2\times L^2\times H^{-1}$ if small initial data $(u_0(x),u_1(x),n_0(x),n_1(x))\in(H^1\times L^2\times L^2\times H^{-1})$. It answers an open problem: Is it globally well-posed in energy space $H^1\times L^2\times L^2\times H^{-1}$ for 3D Klein-Gordon-Zakharov equation with small initial data [1, 2]? The method in this article combines the linear property of the equation (dispersive property) with nonlinear property of the equation (energy inequalities). We mainly extend the spaces $\mathbf{F}^s$ and $\mathbf{N}^s$ in one dimension [3] to higher dimension.

Key words Global well-posedness; 3D Klein-Gordon-Zakharov equation; dyadic $X_{s,b}$ spaces in higher dimension

2010 MR Subject Classification 35E15; 35Q53

1 Introduction

We consider the global well-posedness in the energy space for the Cauchy problem of the Klein-Gordon-Zakharov equations in three dimensional space:

$$\begin{cases} u_{tt}-\Delta u+u=-nu, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}_+,\\ n_{tt}-c^2\Delta n=\Delta|u|^2, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}_+,\\ u(x,0)=u_0(x), \quad \partial_t u(x,0)=u_1(x), \quad n(x,0)=n_0(x), \quad \partial_t n(x,0)=n_1(x), \end{cases} \tag{1.1}$$

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where $0 < c \leq 1$, $x = (\tilde{x}_{*,1}, \tilde{x}_{*,2}, \tilde{x}_{*,3}) \in \mathbb{R}^3$, and Δ is the laplacian in $\mathbb{R}^3$.

The propagation speed in first equation of (1.1) is normalized as unit, while that in second equation of (1.1) is denoted by c. Equations (1.1) describes the interaction of the Langmuir wave and the ion acoustic wave in a plasma (see Dendy [[4]; Chapter 6] and Zakharov [5]). The function u denotes the fast time scale component of electric field raised by electrons and the function n denotes the deviation of ion density from its equilibrium. The functions u and n are originally real vector valued and real scalar valued, respectively. In this article, however, we take two functions u and n as complex scalar valued, because it does not matter what kind of value the functions u and n take in our argument.

The solutions u and n of equations (1.1) formally satisfy the following energy identity:

$$H(u, u_t, n, n_t) = H(u_0, u_1, n_0, n_1), \tag{1.2}$$

where

$$H(u, u_t, n, n_t) = \|u\|_{H^1}^2 + \|u_t\|_{L^2}^2 + \frac{1}{2}c^2\|n\|_{L^2}^2 + \frac{1}{2}\|n_t\|_{H^{-1}}^2 + \mathrm{Re}\int_{\mathbb{R}^3} n(t,x)|u(x,t)|^2 \mathrm{d}x. \tag{1.3}$$

If $0 < c < 1$, then it means that the propagation speed in first equation of (1.1) is about one thousand times as large as that in the second equation of (1.1). In 1999, T. Ozawa, K. Tsutaya, and Y. Tsutsumi [6] used the Bourgian's space to obtain the local well-posedness of (1.1) in energy space since the resonance is weak. By energy (1.2), we can obtain the global well-posedness of (1.1) in energy space for small initial data.

If $c = 1$, there exists strong resonance of equation (1.1). Applying the argument of normal forms to equation (1.1), Ozawa et al [1] proved the existence of global solutions to equation (1.1) for small initial data. However, in [1] one needs high regularity assumptions on the data to ensure the global existence. Tsutaya [2] lowed high regularity assumptions on the data, but the assumptions on the data in [2] also need high regularity, moreover the initial data also belong to the weighted function space. Thus, there exists an open problem: Is it globally well-posed in energy space $H^1 \times L^2 \times L^2 \times H^{-1}$ for 3D Klein-Gordon-Zakharov equation with small initial data [1, 2]?

In this article, we will solve this problem; show that if $c = 1$, then the Cauchy problem (1.1) is globally well-posed in energy space $H^1 \times L^2 \times L^2 \times H^{-1}$ with small initial data.

Now, we would outline the proof of our results. Let

$$\widetilde{\mathbf{u}}_\pm = u \pm \mathrm{i}(1-\Delta)^{1/2}\partial_t u, \quad \widetilde{\mathbf{n}}_\pm = n \pm \mathrm{i}(1-\Delta)^{1/2}\partial_t n.$$

Then, the Cauchy problem (1.1) is rewritten as follows:

$$\begin{cases} \mathrm{i}\partial_t\widetilde{\mathbf{u}}_\pm \mp (1-\Delta)^{1/2}\widetilde{\mathbf{u}}_\pm = \pm\frac{1}{4}(1-\Delta)^{-1/2}(\widetilde{\mathbf{n}}_+ + \widetilde{\mathbf{n}}_-)(\widetilde{\mathbf{u}}_+ + \widetilde{\mathbf{u}}_-), \\ \mathrm{i}\partial_t\widetilde{\mathbf{n}}_\pm \mp (1-\Delta)^{1/2}\widetilde{\mathbf{n}}_\pm = \pm\frac{1}{4}(1-\Delta)^{-1/2}\Delta|\widetilde{\mathbf{u}}_+ + \widetilde{\mathbf{u}}_-|^2 \mp \frac{1}{2}(1-\Delta)^{-1/2}(\widetilde{\mathbf{n}}_+ + \widetilde{\mathbf{n}}_-), \\ \widetilde{\mathbf{u}}_\pm(x,0) = \widetilde{\mathbf{u}}_{\pm 0}(x), \quad \widetilde{\mathbf{n}}_\pm(x,0) = \widetilde{\mathbf{n}}_{\pm 0}(x); \end{cases} \tag{1.4}$$

where

$$\widetilde{\mathbf{u}}_{\pm 0}(x) = u_0 \pm (1-\Delta)^{1/2}u_1, \quad \widetilde{\mathbf{n}}_{\pm 0}(x) = n_0 \pm (1-\Delta)^{1/2}n_1.$$

Note

$$(\widetilde{\mathbf{u}}_{\pm 0}(x), \widetilde{\mathbf{n}}_{\pm 0}(x)) \in H^1 \times L^2.$$