Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2016,**36B**(3):655–665
Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

SOME RESULTS ON CONTROLLED FRAMES IN HILBERT SPACES*

Kamran MUSAZADEH† *Hassan KHANDANI*
Department of Mathematics, College of Science, Mahabad Branch,
Islamic Azad University, Mahabad, Iran
E-mail: kamran_ms2004@yahoo.com; khandani.hassan@yahoo.com

Abstract We use two appropriate bounded invertible operators to define a controlled frame with optimal frame bounds. We characterize those operators that produces Parseval controlled frames also we state a way to construct nearly Parseval controlled frames. We introduce a new perturbation of controlled frames to obtain new frames from a given one. Also we reduce the distance of frames by appropriate operators and produce nearly dual frames from two given frames which are not dual frames for each other.

Key words frames; Parseval frames; dual frames; controlled frames; reconstruction formula; perturbation

2010 MR Subject Classification 94A12; 42C15; 68M10; 46C05

1 Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [15] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossman, and Meyer [14] and popularized from then on. Today, frame theory has an abundance of applications in pure mathematics, applied mathematics, engineering, medicine and even quantum communication. We refer to [4, 6, 7, 11, 22] for an introduction to frame theory and its applications. With the deep development of frame theory, some various generalizations of frames were given by some authors, such as bounded quasi-projectors [16], pseudo-frames [19], frames of subspaces (fusion frames) [1, 8, 18], oblique frames [12], etc. In 2006, a new generalization of the frame named g-frame was introduced by Sun [23, 24] in a complex Hilbert space. G-frames are natural generalizations of frames which cover the above generalizations of frames.

Controlled frames for spherical wavelets were introduced in [5] to get a numerically more efficient approximation algorithm and the related theory for general frames were developed in [3]. In this paper we characterize all operators which produces Parseval frames from a usual frame. Also we introduce all such operators that make nearly Parseval controlled frames which may be useful objects specially for infinite dimensional Hilbert spaces.

Throughout this paper H is a separable Hilbert space , and $GL(H)$ denotes the set of all bounded linear operators which have bounded inverses. It is easy to see that if $S, T \in GL(H)$,

*Received June 25, 2015. Supported by IAU-Mahabad branch (51663931105001).
†Corresponding author: Kamran MUSAZADEH

then T^*, T^{-1} and ST are also in $GL(H)$. Let $GL^+(H)$ be the set of all positive operators in $GL(H)$. Also Id_H denotes the identity operator on H, $\mathbb{R}$ is the set of real numbers. We denote by $\mathcal{F}$ a sequence of vectors $\{f_i\}_{i\in I}$ (I is a countable index set) in a separable Hilbert space H.

A sequence $\mathcal{F}$ in H is called a frame for H, if there exist constants $0 < C \leq D < \infty$ (lower and upper frame bounds) such that

$$C\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq D\|f\|^2, \qquad \forall f \in H.$$

If $C = D$, then $\mathcal{F}$ is called a C-tight frame, and if $C = D = 1$, it is called a Parseval frame. A Bessel sequence $\mathcal{F}$ is only required to fulfill the upper frame bound estimate but not necessarily the lower estimate.

For a Bessel sequence $\mathcal{F}$ of elements in H we define a linear operator $\theta_{\mathcal{F}} : H \to \ell_2(I)$ by

$$\theta_{\mathcal{F}} f = \{\langle f, f_i\rangle\}_{i\in I}, \qquad \forall f \in H.$$

If $\mathcal{F}$ is a frame for H, then $\theta_{\mathcal{F}}$ is a bounded operator and is called the analysis operator of the frame. The synthesis operator is $\theta^*_{\mathcal{F}}$ and satisfies

$$\theta^*_{\mathcal{F}} : \ell_2(I) \to H, \qquad \theta^*_{\mathcal{F}}\big(\{a_i\}_{i\in I}\big) = \sum_{i\in I} a_i f_i, \qquad \forall \{a_i\}_{i\in I} \in \ell_2(I).$$

It is easy to see that $\mathcal{F}$ is a frame for H with frame bounds C, D if and only if

$$C\|f\|^2 \leq \|\theta_{\mathcal{F}} f\|_2^2 \leq D\|f\|^2, \qquad \forall f \in H.$$

The frame operator $S_{\mathcal{F}} f = \theta^*_{\mathcal{F}}\theta_{\mathcal{F}} f = \sum\limits_{i\in I}\langle f, f_i\rangle f_i$ associated with $\mathcal{F}$ is a bounded, invertible, and positive operator on H. This provides the reconstruction formulas

$$f = S^{-1}_{\mathcal{F}} S_{\mathcal{F}} f = \sum_{i\in I}\langle f, f_i\rangle S^{-1}_{\mathcal{F}} f_i = \sum_{i\in I}\langle f, S^{-1}_{\mathcal{F}} f_i\rangle f_i.$$

Furthermore, $CId_H \leq S_{\mathcal{F}} \leq DId_H$.

2 Controlled Frames

As stated before, controlled frames with one operator as controller operator were first introduced in [3]. Also controlled g-frames with two controller operators were studied in [21]. To get a large class of controlled g-frames it is important to use of two operators. So in this section we introduce controlled frames with two controller operators which is a generalization of controlled frames with one controller operator. Also we express the relation between a frame and a controlled frame.

Definition 2.1 Let $\mathcal{F}$ be a family of vectors in a Hilbert space H. Let $T, U \in GL(H)$. Then $\mathcal{F}$ is called a frame controlled by T and U or (T, U)-controlled frame if there exist two constants

$$0 < C_{TU} \leq D_{TU} < \infty$$

such that

$$C_{TU}\|f\|^2 \leq \sum_{i\in I}\langle f, Tf_i\rangle\langle Uf_i, f\rangle \leq D_{TU}\|f\|^2, \qquad \forall f \in H.$$

We call $\mathcal{F}$ a Parseval (T, U)-controlled frame if $C_{TU} = D_{TU} = 1$. If only the right inequality hold, then we call $\mathcal{F}$ a (T, U)-controlled Bessel sequence.