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NEW SUBCLASS OF ANALYTIC FUNCTIONS IN CONICAL DOMAIN ASSOCIATED WITH A LINEAR OPERATOR*

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Abstract The main object of the present paper is to investigate a number of useful properties such as inclusion relations, distortion bounds, coefficient estimates, subordination results, the Fekete-Szegö problem and some other for a new subclass of analytic functions, which are defined here by means of linear operator. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

Key words analytic functions; subordination; functions with positive real part; linear operators; conic domains

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1 Introduction and Definitions

Let $\mathcal{A}$ be the class of functions having form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{E} = \{z \in \mathbb{C} : |z| < 1\}$. Further, we denote the class $\mathcal{S}$ of all functions in $\mathcal{A}$ which are univalent in $\mathcal{E}$. Let $\mathcal{S}^*(\beta)$, $\mathcal{C}(\beta)$, $\mathcal{K}(\beta)$ and $\mathcal{C}^*(\beta)$ denote the classes of starlike, convex, close-to-convex and quasi-convex of order β, respectively. For details see [1]. Goodman [2] introduced the class $\mathcal{UCV}$ of uniformly convex functions. A function $f(z) \in \mathcal{C}(0)$

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is in the class $\mathcal{UCV}$ if for every circular arc $\xi \subset \mathcal{E}$, with center in $\mathcal{E}$, the arc $f(\xi)$ is convex. An interesting characterization of class $\mathcal{UCV}$ was given in [3], see also [4], as

$$f(z) \in \mathcal{UCV} \ \Leftrightarrow f(z) \in \mathcal{A} \ \text{ and } \ \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{E}).$$

In [5], see also [6], it was introduced the class k-uniformly convex functions, $k \geq 0$, denoted by $k - \mathcal{UCV}$ and the class $k - \mathcal{ST}$ related to $k - \mathcal{UCV}$ by Alexandar type relation, i.e., $f(z) \in k - \mathcal{UCV} \Leftrightarrow zf'(z) \in k - \mathcal{ST}$, where

$$f(z) \in k - \mathcal{UCV} \Leftrightarrow f(z) \in \mathcal{A} \ \text{ and } \ \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathcal{E}).$$

In [5] and [6], respectively, their geometric definitions and connections with the conic domains were also considered. If $k \geq 0$, then the class $k - \mathcal{UCV}$ is defined purely geometrically as a subclass of univalent functions which map the intersection of $\mathcal{E}$ with any disk centered at $\zeta, |\zeta| \leq k$, onto a convex domain. Therefore, the notion of k-uniform convexity is a generalization of the notion of convexity. Observe that, if $k = 0$ then the center ζ is the origin and the class $k - \mathcal{UCV}$ reduces to the class $\mathcal{C}$. Moreover for $k = 1$ it coincides with the class of uniformly convex functions $\mathcal{UCV}$ introduced by Goodman [2] and studied extensively by Rønning [4] and independently by Ma and Minda [3]. We note that the class $k - \mathcal{UCV}$ started much earlier in papers [7, 8] with some additional conditions but without the geometric interpretation.

We say that a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{S}^*_{k,\gamma}$, $k \geq 0$, $\gamma \in \mathbb{C} \setminus \{0\}$ if and only if

$$\mathfrak{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > k\left|\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| \quad (z \in \mathcal{E}).$$

A lot of the author's investigated the properties of the class $\mathcal{S}^*_{k,\gamma}$ and their generalizations in several directions e.g. see [4, 6, 9–12].

If $f(z)$ and $g(z)$ are analytic in $\mathcal{E}$, we say that $f(z)$ is subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, which is analytic in $\mathcal{E}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Furthermore, if the function $g(z)$ is univalent in $\mathcal{E}$, then we have the following equivalence, see [1, 13].

$$f(z) \prec g(z) \quad (z \in \mathcal{E}) \Longleftrightarrow f(0) = g(0) \ \text{ and } \ f(\mathcal{E}) \subset g(\mathcal{E}).$$

For two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ \text{ and } \ g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathcal{E}),$$

the convolution (Hadamard product) of $f(z)$ and $g(z)$ is defined as

$$f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Now, we recall the incomplete beta function $\phi(a, c; z)$ defined by

$$\phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, \tag{1.2}$$

where $(x)_n$ denotes the Pochhammer symbol which is defined in term of Gamma function Γ as:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{for } n = 0, \quad x \neq 0, \\ x(x+1)\cdots(x+n-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \cdots\}. \end{cases} \tag{1.3}$$