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GROUND STATE TRAVELLING WAVES IN INFINITE LATTICES*

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Abstract In this paper, we consider FPU lattices with particles of unit mass. The dynamics of the system is described by the infinite system of second order differential equations

$$\ddot{q}_n = U'(q_{n+1} - q_n) - U'(q_n - q_{n-1}), \quad n \in \mathbb{Z},$$

where q_n denotes the displacement of the n-th lattice site and U is the potential of interaction between two adjacent particles. Inspired by previous work due to Szulkin and Weth (Ground state solutions for some indefinite variational problems, J. Funct. Anal., 257 (2009), 3802-3822), we investigate the existence of solitary ground waves, i.e., nontrivial solutions with least possible energy.

Key words FPU lattice; ground state wave; Nehari manifold

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1 Introduction and Main Results

In this paper, we consider FPU lattices with particles of unit mass. The dynamics of the system is described by the infinite system of second order differential equations

$$\ddot{q}_n = U'(q_{n+1} - q_n) - U'(q_n - q_{n-1}), \quad n \in \mathbb{Z}, \tag{1.1}$$

where q_n denotes the displacement of the n-th lattice site and U is the potential of interaction between two adjacent particles. And equations (1.1) form an infinite system of ordinary differential equations which is a Hamiltonian system with the Hamiltonian

$$H = \sum_{n=-\infty}^{\infty} \Big(\frac{1}{2}p_n^2 + U(q_{n+1} - q_n)\Big),$$

where $p_n = \dot{q}_n$ is the momentum of the n-th particle.

Travelling wave with speed c is a solution of the form

$$q_n(t) = u(n - ct), \quad n \in \mathbb{Z},$$

where $u(t)$, $t \in \mathbb{R}$, is called the profile function. Substituting it in (1.1), we obtain the following equation for the profile function

$$c^2 u'' = U'(u(t+1) - u(t)) - U'(u(t) - u(t-1)). \tag{1.2}$$

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The function $r(t) = u(t+1) - u(t)$ is called the relative displacement profile. A travelling wave is a solitary wave if its relative displacement profile $r(t)$ (equivalently, $u'(t)$) vanishes at infinity.

The study of systems of type (1.1) by using variational methods is quite recent. One of the first results is due to Frieseke and Wattis [4], and the approach is based on appropriate constrained minimization and on the concentration compactness method introduced in Lions [6]. Based on the mountain pass theorem without (PS) condition, Smets and Willem [11] also obtained the existence of solitary waves with prescribed speed. Another approach to this problem was introduced in Pankov and Pflüger [8] and relied on a combination of periodic approximations method and standard mountain pass theorem. In addition, it gives a local convergence of periodic waves to solitary ones and obtains ground waves. In [10], Pankov and Rothos discussed a similar problem with saturable nonlinearities by the same method.

All through this paper, we assume that U is of the form

$$U(r) = \frac{a}{2}r^2 + V(r), \quad a \in \mathbb{R}.$$

It is clear that solutions of (1.2) can be obtained as critical points of the action function

$$J(u) = \int_{-\infty}^{+\infty} \left[\frac{c^2}{2}(u')^2 - U(u(t+1) - u(t))\right] \mathrm{d}t$$

on some appropriate Hilbert space X to be specified later.

Our main assumptions on the potential are as follows

(V_1) V is C^1 on $\mathbb{R}$, $V(0) = V'(0) = 0$ and $V'(r) = o(|r|)$ as $r \to 0$;

(V_2) $\frac{V(r)}{r^2} \to +\infty$ as $|r| \to +\infty$;

(V_3) $r \mapsto \frac{V'(r)}{|r|}$ strictly increasing on $(-\infty, 0)$ and on $(0, +\infty)$.

The following Ambrosetti-Rabinowitz condition introduced in [1] is widely used in many papers treating the superquadratic case.

(AR) There exists a constant $\mu > 2$ such that

$$0 < \mu V(r) \le V'(r)r, \quad r \neq 0.$$

It is well known that (AR) implies (V_2), so it is a stronger condition than (V_2). There are many functions, such as $V(r) = |r|^2 \ln(1+|r|^2)$, which do not satisfy (AR), but satisfy (V_2). A crucial role that (AR) plays is to ensure the boundedness of Palais-Smale sequences.

In this paper, we consider solitary ground waves, i.e., nontrivial solitary travelling waves with profile $u \in X$ such that the critical value $J(u)$ is minimal possible among all nontrivial critical values of J. Since the functional J does not satisfy the Palais-Smale condition, the existence of such waves is not trivial and to prove it we employ the Nehari manifold approach. In [9], Pankov imposed the following condition on the potential,

(N) $V \in C^2$ and there exists $\mu \in (0, 1)$ such that

$$0 < r^{-1}V'(r) \leqslant \mu V''(r), \quad r \neq 0.$$

He first showed that the Nehari manifold is a C^1-manifold and thus a natural constraint for J in the sense that any critical point of J constrained on the Nehari manifold is a critical point of J. Then he verified that the ground value is a mountain pass value, and derived the solitary ground travelling wave via a periodic approximation method. Since we are not assuming V is of class C^2, the Nehari manifold is not a C^1-submanifold of X and hence, we cannot apply