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PERIODIC OPTIMAL CONTROL PROBLEMS GOVERNED BY SEMILINEAR PARABOLIC EQUATIONS WITH IMPULSE CONTROL*

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Abstract This paper is concerned with periodic optimal control problems governed by semilinear parabolic differential equations with impulse control. Pontryagin's maximum principle is derived. The proofs rely on a unique continuation estimate at one time for a linear parabolic equation.

Key words Pontryagin's maximum principle; impulse control; semilinear parabolic equation

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1 Introduction

Let T be a positive number and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with boundary $\partial\Omega$ of class C^2. Let $\tau \in (0, T)$ and ω be a nonempty open subset of Ω. Write χ_ω for the characteristic function of ω. Consider the following semilinear parabolic equation with impulse control:

$$\begin{cases} \partial_t y_1 - \Delta y_1 + f(x,t,y_1) = 0 & \text{in } \Omega \times (0,\tau), \\ \partial_t y_2 - \Delta y_2 + f(x,t,y_2) = 0 & \text{in } \Omega \times (\tau,T), \\ y_1 = 0 & \text{on } \partial\Omega \times (0,\tau), \\ y_2 = 0 & \text{on } \partial\Omega \times (\tau,T), \\ y_2(\tau) = y_1(\tau) + \chi_\omega u & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u \in L^2(\Omega)$. Throughout this paper, we assume that $f : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ satisfies:

(H_1) For each $r \in \mathbb{R}$, $f(\cdot,\cdot,r)$ is a measurable function in $\Omega \times (0,T)$. $f(\cdot,\cdot,0) \in L^2(\Omega \times (0,T))$.

(H_2) For a.e. $(x,t) \in \Omega \times (0,T)$, $f'_r(x,t,\cdot)$ is continuous. Moreover, there exists a positive constant L so that

$$|f'_r(x,t,r)| \leq L \text{ for a.e. } (x,t) \in \Omega \times (0,T) \text{ and } r \in \mathbb{R}.$$

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Denote

$$Y \triangleq W^{1,2}(0,\tau;H^{-1}(\Omega)) \cap L^2(0,\tau;H_0^1(\Omega)) \times W^{1,2}(\tau,T;H^{-1}(\Omega)) \cap L^2(\tau,T;H_0^1(\Omega)).$$

It is well known that for each $u \in L^2(\Omega)$ and $y_0 \in L^2(\Omega)$, (1.1) has a unique solution

$$(y_1(\cdot;y_0), y_2(\cdot;y_0,u)) \in Y \subset C([0,\tau];L^2(\Omega)) \times C([\tau,T];L^2(\Omega))$$

satisfying the initial condition $y_1(0;y_0) = y_0$.

Consider the cost functional J: $Y \times L^2(\Omega) \to \mathbb{R}^+ \triangleq [0,+\infty)$, defined by

$$J(y_1,y_2,u) \triangleq \int_0^\tau g(t,y_1(t))\mathrm{d}t + \int_\tau^T g(t,y_2(t))\mathrm{d}t + \frac{1}{2}\|u\|^2_{L^2(\Omega)},$$

where we assume that

(H$_3$) The functional g: $[0,T] \times L^2(\Omega) \to \mathbb{R}^+$ is measurable in t, $g(\cdot,0) \in L^2(0,T)$ and for every $\delta > 0$, there exists a $C_\delta > 0$ so that

$$|g(t,y) - g(t,z)| \le C_\delta \|y - z\|_{L^2(\Omega)}, \quad \forall\, t \in [0,T],$$
$$\|y\|_{L^2(\Omega)} + \|z\|_{L^2(\Omega)} \le \delta.$$

In this paper, we shall study the following optimal control problem:

$$\text{(P)} \qquad \inf J(y_1,y_2,u)$$

over all $(y_1,y_2,u) \in Y \times L^2(\Omega)$, where (y_1,y_2,u) satisfies equation (1.1) and the state constraint condition $y_1(0) = y_2(T)$.

The main result of this paper is as follows.

Theorem 1.1 Suppose that (H$_1$), (H$_2$) and (H$_3$) hold. Let (y_1^*, y_2^*, u^*) be optimal for problem (P). Then there exists $p \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ so that

$$\begin{cases} \partial_t p + \Delta p - f'_y(x,t,y^*)p \in \partial g(t,y^*) & \text{in } \Omega \times (0,T), \\ p = 0 & \text{on } \partial\Omega \times (0,T), \\ p(0) = p(T) & \text{in } \Omega \end{cases}$$

and

$$\chi_\omega p(\tau) = u^*,$$

here

$$y^*(t) \triangleq \begin{cases} y_1^*(t), & t \in [0,\tau), \\ y_2^*(t), & t \in [\tau,T] \end{cases}$$

and $\partial g(t,y^*)$ denotes the generalized derivative to the second variable at y^* in the sense of Clarke (see page 27 in [1]).

Remark 1.2 When

$$f(x,t,r) = a(x,t)r \ \text{ and } \ \|a\|_{L^\infty(\Omega\times(0,T))} < \lambda_1 \triangleq \inf_{\substack{y \in H_0^1(\Omega) \\ y \neq 0}} \frac{\|\nabla y\|_{L^2(\Omega)}}{\|y\|_{L^2(\Omega)}},$$

we can easily check that for each $u \in L^2(\Omega)$, there exists a unique $z_0 \in L^2(\Omega)$ so that $y_1(0;z_0) = y_2(T;z_0,u)$.