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PARTIAL SCHAUDER ESTIMATES FOR A SUB-ELLIPTIC EQUATION*

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Abstract In this paper, we establish the partial Schauder estimates for the Kohn Laplace equation in the Heisenberg group based on the mean value theorem, the Taylor formula and a priori estimates for the derivatives of the Newton potential.

Key words partial Schauder estimates; Kohn Laplace equation; Heisenberg group

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1 Introduction

In the Euclidean space, Schauder estimates for elliptic and parabolic equations were well studied in [5, 6, 19, 22], etc., which play an important role in the theory of partial differential equations. In brief, if $u \in C^2$ is a solution of $\Delta u = f$, then one can have the estimates for the modulus of D^2u when f is Hölder continuous. The partial Schauder estimates for the solutions of elliptic equations in Euclidean spaces can be derived under incomplete Hölder continuity assumptions, see [9, 11, 20]. Here, the partial Schauder estimates means that the partial derivatives of the solution in some directions are Hölder continuous but fail in others. One of the motivations of this paper is to study the phenomenon about the partial Schauder estimates for sub-elliptic equations.

Some research was done for the Schauder estimates of the operators structured on the non-abelian vector fields. Capogna and Han [7] showed the pointwise Schauder estimates for the operator $L = \sum\limits_{i,j=1}^{m} a_{ij}(x)X_iX_j$ in the Carnot group, where $\{X_j's, j = 1, \cdots, m\}$ span the first layer of the Lie algebra of a Carnot group in $\mathbb{R}^N$. Then, Gutiérrez and Lanconelli [18] considered the Schauder estimates for a class of more general sub-elliptic equations $L = \sum\limits_{i,j=1}^{m} a_{ij}(x)X_iX_j + X_0$ by using the Taylor formula. Bramanti and Brandolini [3] gave the Schauder estimates for

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the operators $a_{ij}X_iX_j - \partial_t$ with X_i satisfying Hörmander's rank condition, by making use of the properties of the fundamental solution of the frozen operator. Capogna showed the C^α regularity of Quasi-linear equations in the Heisenberg group (denoted by $\mathbb{H}^n$) [4], which is the simplest non-abelian nilpotent Lie group. The Kohn Laplace equation in the Heisenberg group is a classical sub-elliptic equaion. The second author and his collaborator derived the Schauder estimates to the Kohn Laplace equation

$$\Delta_{\mathbb{H}^n} u = f \quad \text{in } B_1(0), \tag{1.1}$$

where $B_1(0) = \{\eta \in \mathbb{H}^n, d(\eta, 0) < 1\}$ is the unit ball in the Heisenberg group $\mathbb{H}^n$ and f is Dini continuous. However, only very few results were obtained on the partial Schauder estimates for sub-elliptic equations. One of the purposes of this paper is to extend the results in [10] to the case of partial Schauder estimates under incomplete Dini continuity assumptions.

Let $\xi := (\xi_1, \xi_2, \cdots, \xi_{2n}, t) \in \mathbb{H}^n$ be a given point, and η_m be a given unit vector in the plane

$$P_m := L\{\xi_m, t\} \quad \text{which is generated by} \quad \{\xi_m, t\}. \tag{1.2}$$

Analogously to the definition of Dini continuous in one direction [20], we say that f is Dini continuous in P_m $(m = 1, 2, \cdots, 2n)$ if

$$\int_0^1 \frac{\omega_{f,m}(r)}{r} \mathrm{d}r < \infty,$$

where

$$\omega_{f,m}(r) = \sup\{|f(\xi) - f(\xi \circ t\eta_m)|m, \quad \text{where } \xi, \xi \circ t\eta_m \in B_1, \eta_m \in P_m, |t| < r\}. \tag{1.3}$$

This definition of Dini continouous for a function in a plane can be regarded as an extension from the classical concept of Dini continouous. Indeed, since the space $\mathbb{H}^n$ can be spanned by a collection of planes $\{P_1, P_2, \cdots, P_{2n}\}$, there exists a point $\eta_m \in P_m$ for each $m\,(= 1, 2, \cdots, 2n)$ such that $\xi \circ \eta^{-1} = \xi \circ \eta_1^{-1} \circ (\eta_1 \circ \eta_2^{-1}) \circ \cdots (\eta_i \circ \eta_{i+1}^{-1}) \circ \cdots (\eta_{2n-1} \circ \eta_{2n}^{-1}) \circ \eta_{2n} \circ \eta^{-1}$. Hence,

$$\omega(r) = \sup_{\xi, \eta \in B_1 \ d(\xi,\eta) < r} |f(\xi) - f(\eta)| \le \sum_{m=1}^{2n} \omega_{f,m}(r), \tag{1.4}$$

which means that $\int_0^1 \frac{\omega(t)}{t} \mathrm{d}t < \infty$ (f is Dini continuous) when f is Dini continuous in the plane P_m for all $m = 1, 2, \cdots, 2n$.

In this paper, we assume that the solution is smooth, for example $u \in C^3(B_1)$. By approximation, the following estimates hold for weak solutions.

Theorem 1.1 Let u be a solution of (1.1). If f is Dini continuous in a plane P_m, then $\forall \xi, \eta \in B_{\frac{1}{4}}(0)$,

$$|Z_iZ_mu(\xi) - Z_iZ_mu(\eta)| \le C_Q \left[d \sup_{B_1} |u| + d \sup_{B_1} |f| + \int_0^d \frac{w_{f,m}(t)}{t} \mathrm{d}t + d \int_d^1 \frac{w_{f,m}(t)}{t^2} \mathrm{d}t \right], \tag{1.5}$$

$$|Z_mZ_iu(\xi) - Z_mZ_iu(\eta)| \le C_Q \left[d \sup_{B_1} |u| + d \sup_{B_1} |f| + \int_0^d \frac{w_{f,m}(t)}{t} \mathrm{d}t + d \int_d^1 \frac{w_{f,m}(t)}{t^2} \mathrm{d}t \right], \tag{1.6}$$

where $Z_i \in \{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ are the horizontal gradient operators on $\mathbb{H}^n$, and $d = d(\xi, \eta) = \|\eta^{-1} \circ \xi\|$, $C_Q > 0$ depends on Q.