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EXISTENCE AND UNIQUENESS OF ENTROPY SOLUTION TO PRESSURELESS EULER SYSTEM WITH A FLOCKING DISSIPATION*

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Abstract We study the existence and uniqueness problem for the nonhomogeneous pressureless Euler system with the initial density being a Radon measure. Our uniqueness result is obtained in the same space as the existence theorem. Besides, by counterexample we prove that Huang-Wang's energy condition is also necessary for our nonhomogeneous system.

Key words pressureless Euler system; Cucker-Smale model; entropy solution; flocking

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1 Introduction

In the present paper, we study the following nonhomogeneous pressureless Euler system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x = -\rho u \end{cases} \tag{1.1}$$

with initial data

$$0 \le \rho_0 \in M_{\rm loc}(\mathbb{R}), \quad u_0(x) \in L^\infty(\mathbb{R}),$$

where ρ and u are density and velocity, respectively.

This system is derived from Cucker-Smale model, which is used to describe flocking phenomenon. The word flocking represented the phenomenon that autonomous agents reach a consensus state based on limited environmental information and simple rules. The study of flocking based on mathematical models was first started from the work of Vicsek et al. [17], and was further motivated by the hydrodynamic approach [16]. Our system (1.1) can be viewed as a close to equilibrium model for the hydrodynamic Cucker-Smale model [5, 9]. For the detailed derivation, we refer the reader to [8, 11].

Recently, Ha, Huang and Wang [8] studied this problem with the initial density $\rho_0 \in L_{\rm loc}(\mathbb{R})$ and $\rho_0 > 0$ a.e. in $\mathbb{R}$. However, this is not natural since the solution to this system is measure in general. So we should consider this problem in measure space rather than in function space

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for the general case. In this paper, we will settle this problem and our uniqueness theorem is obtained in the same space, i.e., Radon measure space as the existence theorem.

Now let us review some related work for the homogeneous counterpart of our system (1.1), which reads as

$$\begin{cases}\rho_t + (\rho u)_x = 0,\\ (\rho u)_t + (\rho u^2)_x = 0.\end{cases} \tag{1.2}$$

This model is used in plasma physics and has attracted a great deal of attention in mathematic since its solution is measure in general, which poses new challenge to the analysis of the posedness. For the existence of the global weak solution, the result was first obtained independently by Brenier and Grenier [4] and E et al. [7]. Wang et al. [18] extended their results. Boudin [3] showed that the weak solution can also be obtained as limit of solutions to a viscous pressureless model. For the uniqueness of the global weak solution, the authors of [1, 4] and [7] found that the Lax entropy condition was insufficient to guarantee it. E et al. [7] pointed out that the Oleinik entropy condition might be necessary. Along this line, Wang and Ding [18] proved the uniqueness of the weak solution for the case that the initial density $\rho_0 \in L_{\text{loc}}(\mathbb{R})$ and $\rho_0 > 0$ a.e.. Similar results were also obtained by Bouchut and Jame [2]. As for the general case that ρ_0 is a Radon measure, it is more subtle and difficult. Huang, Wang [14] found that besides the Oleinik entropy condition, it is also important to require the energy to be weakly continuous initially. They called it energy condition and further proved that the energy condition is necessary and sufficient for the uniqueness. Our idea in this paper mainly comes from their famous work.

The strategy is as follows: First, we manage to construct the entropy solution by the potential functional

$$F(y; x, t) = \int_{0+0}^{y-0} [\eta + u_0(\eta)(1 - \mathrm{e}^{-t}) - x]\mathrm{d}m_0(\eta) \tag{1.3}$$

with $m_0(x) = \rho_0([0, x))$. Then we show that any entropy solution coincides with the solution we construct. Thus the uniqueness is achieved. We find that $F(y; x, t)$ only depends on the initial data, which will play a crucial role in our proof. We shall investigate the properties of its minimizer. It is the basis of our proof. When ρ_0 is a Radon measure, we can first prove the uniqueness in the region $t \geq t_1 > 0$ instead of $t \geq 0$. This is because the characteristic curves issued from some points of x-axis may not be unique. Then we study the convergence of the sequence (ρ_{t_1}, u_{t_1}), as $t_1 \to 0 + 0$. In order to prove the limit coincides with the entropy solution we construct, we need the energy condition since ρ_0 may have mass on some points of x-axis. Besides, we can show that the energy condition is necessary by counterexample.

The main feature of the pressureless system is the formation of δ-shock no matter how smooth the initial values are. Motivated by the previous work, we define the following potential function $m(x, t)$ by

$$m(x, t) = \int_{(0,0)}^{(x,t)} \rho \mathrm{d}x - \rho u \mathrm{d}t. \tag{1.4}$$

Due to the conserved equation$(1.1)_1$, $m(x, t)$ is independent of the integral path and satisfies