Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2016,**36B**(5):1305–1316
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR A HARMONIC EQUATION WITH CRITICAL NONLINEARITY*

Kamal OULD BOUH
Department of Mathematics, College of Sciences, Taibah University, P.O. Box 30097, Almadinah Almunawwarah, KSA
E-mail: *kamal_bouh@yahoo.fr; hbouh@taibahu.edu.sa*

Abstract This paper is concerned with the harmonic equation $(\mathrm{P}_{\mp\varepsilon}):\ \Delta u=0,\ u>0$ in $\mathbb{B}^n$ and $\frac{\partial u}{\partial \nu}+\frac{n-2}{2}u=\frac{n-2}{2}Ku^{\frac{n}{n-2}\mp\varepsilon}$ on $\mathbb{S}^{n-1}$ where $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$, $n\geq 4$ with Euclidean metric g_0, $\partial\mathbb{B}^n=\mathbb{S}^{n-1}$ is its boundary, K is a function on $\mathbb{S}^{n-1}$ and ε is a small positive parameter. We construct solutions of the subcritical equation $(\mathrm{P}_{-\varepsilon})$ which blow up at one critical point of K. We give also a sufficient condition on the function K to ensure the nonexistence of solutions for $(\mathrm{P}_{-\varepsilon})$ which blow up at one point. Finally, we prove a nonexistence result of single peaked solutions for the supercritical equation $(\mathrm{P}_{+\varepsilon})$.

Key words variational problem; critical points; harmonic equation; mean curvature; critical exponent

2010 MR Subject Classification 35J20; 35J60

1 Introduction

Let us consider the following nonlinear boundary value problem on n-dimensional Riemannian manifold with boundary (M,g), with $n\geq 3$:

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_g u+R_g u=0 & \text{in } \mathring{M},\\ \dfrac{2}{n-2}\dfrac{\partial u}{\partial \nu}+K_g u=cu^{\frac{n}{n-2}} & \text{on } \partial M,\end{cases} \tag{1.1}$$

where $\mathring{M}=M\setminus\partial M$ denotes the interior of M, R_g is the scalar curvature of M, K_g is the mean curvature of ∂M, ν is the outward unit vector with respect to the metric g, and c is a constant whose sign is uniquely determined by the conformal structure. Indeed, if $\bar g=u^{\frac{4}{n-2}}g$, then the metric $\bar g$ has zero scalar curvature and the boundary has a constant mean curvature with respect to $\bar g$.

Escobar studied this problem in [11]. He showed that most compact manifolds with boundary admit such conformally related metrics.

In view of the above equation, it is natural to consider the problem of prescribing boundary mean curvature with zero scalar curvature, that is: given a function $K:\partial M\longrightarrow\mathbb{R}$, does there

*Received June 23, 2015; revised November 17, 2015.

exists a metric g' conformally equivalent to g such that $R_{g'} \equiv 0$ and $K_{g'} \equiv K$? From equation (1.1), the problem is equivalent to finding a smooth positive solution v to the following equation,

$$\begin{cases} -4\dfrac{n-1}{n-2}\Delta_g v + R_g v = 0 & \text{in } \overset{\circ}{M}, \\ \dfrac{2}{n-2}\dfrac{\partial v}{\partial \nu} + K_g v = K v^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases} \tag{1.2}$$

In this article, we are interested in the case where a non compact group of conformal transformations acts on the equation so that Kazdan-Warner type conditions give rise to obstructions as in the Nirenberg problem (see [15]). The simplest situation is the following one.

Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ with Euclidean metric g_0. Its boundary will be denoted by $\partial\mathbb{B}^n = \mathbb{S}^{n-1}$ and will be endowed by the standard metric g_0. Let K be a function on $\mathbb{S}^{n-1}$. In this case, our problem becomes

$$\text{(P)} \qquad \begin{cases} \Delta u = 0 \quad \text{and } u > 0 & \text{in } \mathbb{B}^n, \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2}u = \dfrac{n-2}{2}K u^{\frac{n}{n-2}} & \text{on } \mathbb{S}^{n-1}. \end{cases}$$

Previously, Cherrier [8] studied the regularity question for this equation. He showed that solution of (P) which are of class H^1 are also smooth. In [12], Escobar studied this problem (P) on manifolds which are not equivalent to the standard ball. On the ball, sufficient conditions on K in dimensions 3 and 4 were given in [14], and [9], and a perturbative results were obtained in [7]. In [1], the authors developed a Morse theoretical approach to this problem in the 4-dimensional case providing some multiplicity results under generic conditions on the function K.

Note that the exponent $\frac{2(n-1)}{n-2}$ is critical for the Sobolev trace embedding $H^1(\mathbb{B}^n) \to L^q(\mathbb{S}^{n-1})$. This embedding being not compact. Hence, for the study of problem (P), it is interesting to approach it by the following family of harmonic problems

$$(\mathrm{P}_{\mp\varepsilon}) \qquad \begin{cases} \Delta u = 0 \qquad \text{and } u > 0 & \text{in } \mathbb{B}^n \\ \dfrac{\partial u}{\partial \nu} + \dfrac{n-2}{2}u = \dfrac{n-2}{2}K u^{\frac{n}{n-2}\mp\varepsilon} & \text{on } \mathbb{S}^{n-1}, \end{cases}$$

where ε is a small positive parameter.

Our aim, in this paper, is to give sufficient conditions on K such that problem $(\mathrm{P}_{\mp\varepsilon})$ admits a positive solution. It is easy to see that a necessary condition for solving the problem is that K has to be positive somewhere. Note that some related problems of type $(\mathrm{P}_{\mp\varepsilon})$, in case of bounded domains, were studied in [4–6, 10, 16, 17] and the references therein.

Our first result deals with construction of single peaked solutions for the subcritical harmonic problem $(\mathrm{P}_{-\varepsilon})$ with $\varepsilon > 0$. More precisely, we have

Theorem 1.1 Let $n \geq 4$ and y be a nondegenerate critical point of K with $-\Delta K(y) > 0$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, problem $(\mathrm{P}_{-\varepsilon})$ has a solution (u_ε) of the form

$$u_\varepsilon = \alpha_\varepsilon \tilde{\delta}_{(x_\varepsilon, \lambda_\varepsilon)} + v_\varepsilon \tag{1.3}$$

with $v_\varepsilon \in E_{(x_\varepsilon, \lambda_\varepsilon)}$ and as $\varepsilon \to 0$,

$$\alpha_\varepsilon \to K(y)^{(2-n)/2};\ \|v_\varepsilon\| \to 0;\ x_\varepsilon \to y,\ \lambda_\varepsilon \to +\infty, \tag{1.4}$$