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FRACTIONAL INTEGRAL INEQUALITIES AND THEIR APPLICATIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS*

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Abstract In this paper, first we obtain some new fractional integral inequalities. Then using these inequalities and fixed point theorems, we prove the existence of solutions for two different classes of functional fractional differential equations.

Key words fractional integral inequality; existence of solution; Caputo fractional derivative; fractional differential equation; fixed point

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1 Introduction

Fractional differential equations appear in various fields of engineering and science such as viscoelasticity, electrochemistry, control, electromagnetic, porous media, etc. For example in [1–5], we can see applications of fractional differential equations in signal processing, complex dynamics in biological tissues, viscoelastic materials, thermal systems and heat conduction. One can see the application of fractional differential equations in complex physical systems, physical systems description and control, in [6–8]. In the books of Mainardi [9] and Tarasov [8], there are applications of fractional calculus in complex physical systems and dynamics of viscoelastic. To see some developments of fractional calculus, related to special functions, we refer the reader to the book by Kiryakova [10] (also see [11–13]). Also in the monograph of Klafter et al. [14], we can find the latest developments in the field of fractional dynamics. To see some recent results on the existence of solutions for fractional differential equations, we refer the reader to [15–25]. In the monographs of Kilbas et al. [26], Podlubny [27], Diethelm [28] and Samko et al. [29] there are some basic information and existence results for various type of fractional differential equations.

Integral inequalities are very useful in the study of ordinary differential and integral equations. For example the Gronwall-Bellman inequality and its generalizations play an important role in the discussion of existence, uniqueness, boundedness, and qualitative behavior of solutions (see [30, 31]). Motivated by applications of fractional integral inequalities (see [32, 33]),

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we study the following fractional integral inequalities

$$u(t) \leq p(t) + q(t) \sum_{i=1}^{n} (I_{a+}^{\alpha_i} u)(t), \qquad t \in [a,b], \tag{1.1}$$

$$u(t) \leq p(t) + q(t) \sum_{i=1}^{n} (I_{0+}^{\alpha_i} u)(\mu_i t), \qquad t \in [0,b], \tag{1.2}$$

where $n \in \mathbb{N}$, $\alpha_i > 0$, $0 < \mu_i \leq 1$ for $i = 1, 2, \cdots, n$, $0 \leq a < b < \infty$ and $I_{a+}^{\alpha_i}$ is the Riemann-Liouville fractional integral operator. Our main results are as follows.

Theorem 1.1 Let u be a nonnegative continuous function defined on $I = [a,b]$, and let $p(t) : I \to (0,\infty)$ be a nondecreasing continuous function. Assume that $q(t) : I \to [0,\infty)$ is a nondecreasing continuous function. If u satisfies inequality (1.1), then for $k \in \mathbb{N}$ such that $(k+1)\min\{\alpha_1, \alpha_2, \cdots, \alpha_n\} > 1$,

$$u(t) \leq P_k(t) \exp\bigg(\int_a^t H_{k+1}(t,s)\mathrm{d}s\bigg), \qquad t \in I, \tag{1.3}$$

where

$$P_k(t) := p(t)\bigg(1 + \sum_{j=1}^{k} q^j(t) \sum_{\substack{i_1+\cdots+i_n=j \\ 0\leq i_1,\cdots,i_n\leq j}} \binom{j}{i_1,\cdots,i_n} \frac{(t-a)^{i_1\alpha_1+\cdots+i_n\alpha_n}}{\Gamma(1+i_1\alpha_1+\cdots+i_n\alpha_n)}\bigg), \tag{1.4}$$

$$H_{k+1}(t,s) := q^{k+1}(t) \sum_{\substack{j_1+\cdots+j_n=k+1 \\ 0\leq j_1,\cdots,j_n\leq k+1}} \binom{k+1}{j_1,\cdots,j_n} \frac{(t-s)^{j_1\alpha_1+\cdots+j_n\alpha_n-1}}{\Gamma(j_1\alpha_1+\cdots+j_n\alpha_n)}. \tag{1.5}$$

Theorem 1.2 Let u be a nonnegative continuous function defined on $J = [0,b]$, and let $p(t) : J \to (0,\infty)$ be a nondecreasing continuous function. Assume that $q(t) : J \to [0,\infty)$ is a nondecreasing continuous function. If u satisfies inequality (1.2), then for $k \in \mathbb{N}$ such that $(k+1)\min\{\alpha_1, \alpha_2, \cdots, \alpha_n\} > 1$,

$$u(t) \leq \mathcal{P}_k(t) \exp\bigg(\int_0^t \mathcal{H}_{k+1}(t,s)\mathrm{d}s\bigg), \qquad t \in J, \tag{1.6}$$

where

$$\mathcal{P}_k(t) := p(t) + \sum_{j=1}^{k} \mu^{-\frac{j(j-1)}{2}\alpha} q^j(t) \times \sum_{\substack{i_1+\cdots+i_n=j \\ 0\leq i_1,\cdots,i_n\leq j}} \binom{j}{i_1,\cdots,i_n} \frac{p(\mu_1^{i_1}\cdots\mu_n^{i_n}t)(\mu_1^{i_1}\cdots\mu_n^{i_n}t)^{i_1\alpha_1+\cdots+i_n\alpha_n}}{\Gamma(1+i_1\alpha_1+\cdots+i_n\alpha_n)}, \tag{1.7}$$

$$\mathcal{H}_{k+1}(t,s) := \mu^{-\frac{k(k+1)\alpha}{2}} H_{k+1}(t,s), \quad \mu = \min\{\mu_1, \mu_2, \cdots, \mu_n\}, \quad \alpha = \max\{\alpha_1, \alpha_2, \cdots, \alpha_n\}, \tag{1.8}$$

and $H_{k+1}(t,s)$ is defined by (1.5).

The Lipschitz continuity of the nonlinear part of a fractional differential equation is a basic assumption considered in many papers [19, 20, 22, 32, 34, 35]. By Theorem 1.1 and a nonlinear alternative of Leray-Schauder type [36], we prove the existence of solution for the following functional fractional differential equation without using the Lipschitz continuity

$$\begin{cases} ({}^{c}\mathcal{D}_{a+}^{\alpha} u)(t) = f\Big(t, u(t), ({}^{c}\mathcal{D}_{a+}^{\alpha_1} u)(t), ({}^{c}\mathcal{D}_{a+}^{\alpha_2} u)(t)\Big), \ t \in (a,b], \\ u(a) = u_0, \end{cases} \tag{1.9}$$