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LIMITING DIRECTION AND BAKER WANDERING DOMAIN OF ENTIRE SOLUTIONS OF DIFFERENTIAL EQUATIONS*

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Abstract In this paper, we mainly investigate entire solutions of complex differential equations with coefficients involving exponential functions, and obtain the dynamical properties of the solutions, their derivatives and primitives. With some conditions on coefficients, for the solutions, their derivatives and their primitives, we consider the common limiting directions of Julia set and the existence of Baker wandering domain.

Key words limiting direction; Baker wandering domain; entire function; linear differential equation

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1 Introduction and Main Results

The Nevanlinna theory is an important tool in this paper, its usual notations and basic results come mainly from [9, 11, 15, 22]. We use $\lambda(f)$ and $\mu(f)$ to denote the order and the lower order of f respectively, which are defined as [22, Definition 1.6]

$$\lambda(f)=\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{\log r},\quad \mu(f)=\liminf_{r\to\infty}\frac{\log^+ T(r,f)}{\log r}.$$

Let $f:\mathbb{C}\to\overline{\mathbb{C}}$ be a transcendental meromorphic function, where $\mathbb{C}$ is the complex plane and $\overline{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. The Fatou set $F(f)$ of $f(z)$ is the subset of $\mathbb{C}$ where the iterates $f^n(z)(n=1,2,\cdots)$ of f are well defined and $\{f^n(z)\}$ forms a normal family. The complement of $F(f)$ is called the Julia set $J(f)$ of $f(z)$. It is well known that $F(f)$ is open, $J(f)$ is closed and non-empty. In general, the Julia set is very complicated. Some basic knowledge of complex dynamics of meromorphic functions can be found in [7, 27].

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For a transcendental entire function $f(z)$, Baker [2] firstly observed that $J(f)$ can not lie in finitely many rays emanating from the origin. Qiao [18] introduced the limiting direction of $J(f)$, which means a limit θ of the set $\{\arg z_n \,|\, z_n \in J(f)\,\text{is an unbound sequence}\}$. For such limit θ, Zheng also say that the Julia set has the radial distribution with respect to the ray $\arg z = \theta$ [23]. Define

$$\Delta(f) = \{\theta \in [0, 2\pi) : \arg z = \theta \text{ is a limiting direction of } J(f)\}.$$

Clearly, $\Delta(f)$ is closed, by $\operatorname{mes}\Delta(f)$ we stand for its linear measure.

If the transcendental entire function $f(z)$ satisfies $\mu(f) < \infty$, then $\operatorname{mes}\Delta(f) \geq \min\{2\pi, \frac{\pi}{\mu(f)}\}$, see [18]. Furthermore, Qiao [19, Theorem 1] investigated the common limiting directions of all f's derivatives and primitives, and his result is stated as follows.

Theorem A Let f be a transcendental entire function of lower order $\mu < \infty$. Then there exists a closed interval $I \subset \mathbb{R}$ such that all $\theta \in I$ are the common limiting directions of $J(f^{(n)})$, $n = 0, \pm 1, \pm 2, \cdots$, and $\operatorname{mes} I \geq \min\{2\pi, \pi/\mu\}$, here $f^{(n)}$ denotes the n-th derivative or the n-th integral primitive of f for $n \geq 0$ or $n < 0$ respectively.

The example in [2] shows that there exists an entire function of infinite lower order whose Julia set has only one limiting direction $\theta = 0$. Later some observations for a transcendental meromorphic function $f(z)$ were made by [20, 23]: if $\mu(f) < \infty$ and $\delta(\infty, f) > 0$, then

$$\operatorname{mes}\Delta(f) \geq \min\left\{2\pi, \frac{4}{\mu(f)} \arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right\}.$$

For a connected component U of $F(f)$, we know that $f^n(U)$ must be contained in a component U_n of $F(f)$. If all U_n are different, then U is called a wandering domain of f. By Sullivan's famous theorem, rational functions have no wandering domains. For transcendental entire functions, it has been shown earlier by Baker [4] that such domains may exists. The wandering domain in Baker's example is multiply connected, then such wandering domain was named after his name. For the convenience of the readers, we still state the definition.

Definition 1 For the wandering domain U, if all U_n are multiply connected component of $F(f)$ which surrounds 0, and the Euclidean distance $\operatorname{dist}(0, U_n) \to +\infty$ as $n \to +\infty$, then U is called Baker wandering domain.

If f is a transcendental entire function, then each multiply connected component of $F(f)$ must be a Baker wandering domain, see [3]. There are some criterions of non-existence of the Baker wandering domains [5, 7], which also determine whether there exists only simply connected Fatou component for given entire functions. For example, there is an interesting result in [5, Corollary].

Theorem B If the entire function f has path to ∞ on which f is bounded, then all components of $F(f)$ are simply-connected.

There exists the radial growth property of the exponential functions $A(z)\mathrm{e}^{P(z)}$, where $P(z)$ is a non-constant polynomial, and $A(z)$ is an entire function of order less than $\deg P$, see Lemma 2.6. Clearly by Theorem B, $A(z)\mathrm{e}^{P(z)}$ only has simply connected Fatou component. Zheng continued Baker and Bergweiler's work, and investigated transcendental meromorphic functions with at most finitely many poles [25].

For differential equations, the solutions are always controlled by the behavior of coefficients. When there is a dominate coefficient A_0 in the sense $T(r, A_j) = o(T(r, A_0))$ $(j = 1, 2, \cdots, n-1)$