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LEVEL SETS AND EQUIVALENCES OF MORAN-TYPE SETS*

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Abstract In the paper, we consider Moran-type sets E_a given by sequences $\{a_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$. we prove that E_a may be decompose into the disjoint union of level sets. Moreover, we define three type of equivalence between two dimension functions associated to two Moran-type sets, respectively, and we classify Moran-type sets by these equivalent relations.

Key words Moran-type sets; dimension function; level set; logarithmical equivalence

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1 Introduction

Fractal sets may be obtained by removing a sequence of disjoint regions from a given set, which are named cut-out sets. Obviously, all compact subsets in $\mathbb{R}$ can be obtained in this manner. For example, the middle-third Cantor set in $\mathbb{R}$. Similarly, in the plane, the Sierpinski triangle is obtained by removing a sequence of equilateral triangles from an initial equilateral triangle (see [6]).

Let A be a compact interval in $\mathbb{R}$ and $\{A_k\}_{k=1}^{\infty}$ be the disjoint open subintervals of A with $|A| = \sum\limits_{k=1}^{\infty} |A_k|$, where $|A_k|$ denotes the length of A_k. Let $E = A \backslash (\bigcup\limits_{k=1}^{\infty} A_k)$. Then E is a compact set with Lebesgue measure zero and $\bigcup\limits_{k\geq 1} A_k = E^c \cap A$. We call E a cut-out set and A_k the

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complementary intervals. Clearly the set E is determined by the intervals A_k, some geometric information can be obtained from only knowing the lengths and independent of the position of A_k. Note that cut-out sets are strongly connected with gap sequences of fractals which often characterise the geometric properties of fractal sets, and we refer the readers to [4, 21] for details.

Let $a=\{a_n\}_{n=1}^{\infty}$ be a positive non-increasing sequence with $\sum a_n$ convergent. There are many possible compact sets E with complementary intervals of lengths $\{a_n\}$, and let $\mathscr{C}_a$ be the collection of all such cut-out sets generated by a. Besicovitch and Taylor in [1] studied the Hausdorff dimension of these cut-out sets and proved that $\{\dim_H E; E\in\mathscr{C}_a\}=[0,\alpha(a)]$, where $\alpha(a)=\liminf\limits_{n\to\infty}\alpha_n(a)$, and $\alpha_n(a)$ satisfies $n\big(\frac{\sum_{k\geq n}a_k}{n}\big)^{\alpha_n(a)}=1$. Moreover in [27], Xiong and Wu showed that $\mathscr{C}_a$ is a compact metric space with respect to the Hausdorff distance ρ and studied density-type properties in $(\mathscr{C}_a,\rho)$. There is a special class of such cut-out sets studied in literatures, called Cantor set C_a, where $a=\{a_k\}_{k=1}^{\infty}$ is a sequence with $\sum\limits_{k=1}^{\infty}a_k=1$ and the initial set $[0,1]$. At each step, remove an open interval from each remained closed interval, the lengths are arranged by the order of a_k from left to right. Dimension properties were studied by many authors under various restrictions, see [1, 2, 7].

In this paper, we will study another class of cut-out sets which is a generalization of C_a. Let $\{a_k\}_{k\geq1}$ be a positive non-increasing sequence with $\sum\limits_{k\geq1}a_k$ convergent and $\{n_k\}$ be an integer sequence. Given the initial set $I=\Big[0,\sum\limits_{k\geq1}a_k\Big]$. We first remove n_1-1 open intervals from I with lengths $a_1,a_2,\cdots,a_{n_1-1}$ from left to right, and there remain n_1 intervals, indexed by $I_1,\ I_2,\cdots,I_{n_1}$. On the second step, we remove n_2-1 open intervals from each I_j $(1\leq j\leq n_1)$ with lengths $a_{n_1+(n_2-1)(j-1)+1},\ \cdots,\ a_{n_1+(n_2-1)(j-1)+n_2-2}$, and there remain n_2 closed subsets, indiced by $I_{j1},I_{j2},\cdots,I_{jn_2}$. Suppose $I_{i_1i_2\cdots i_k}$ is the j-th interval in step k, we remove $n_{k+1}-1$ open intervals from interval $I_{i_1i_2\cdots i_k}$ with lengths $a_{n_1n_2\cdots n_k+(n_{k+1}-1)(j-1)+1},\cdots,$ $a_{n_1n_2\cdots n_k+(n_{k+1}-1)(j-1)+n_{k+1}-1}$. Continue this process, we call the limit set

$$E_{a,\{n_k\}}=\bigcap_{k=1}^{\infty}\bigcup_{i_1i_2\cdots i_k}I_{i_1i_2\cdots i_k}$$

Moran-type set which is compact and perfect.

Note that the construction uniquely determines the location of gap at each step. For instance, the location of the first gap in Cantor set C_a is determined by the interval whose length is equal to $a_2+a_4+a_5+a_8+\cdots$. Such Moran-type sets can be thought as generalized Moran sets (see [11]) and need not be central or (quasi-)self-similar. When $\{n_k\}\equiv2$, $E_{a,\{n_k\}}$ is the Cantor set C_a defined in [1]. Due to the sequence $\{n_k\}_{k\geq1}$, the structure of $E_{a,\{n_k\}}$ is more general and more complicated than C_a. For example, the lengths of intervals in C_a have "decreasing property" in some sense, that is, for each $k\geq1$, the lengths of all intervals in kth step are longer than in $(k+1)$th step. But this "decreasing property" does not hold for $E_{a,\{n_k\}}$ if $\{n_k\}$ is not a constant sequence.

Furthermore, we will explore the level sets of E_a and the complement of the union of level sets. Since level sets of E_a may be viewed as a partial decomposition of the measure μ into a family of subfractals, it is also an important topic in Fractal geometry, see [11, 26] for further information. The complement of the set $\bigcup\limits_{\alpha>0}\varepsilon(\mu,\alpha)$ is the collection of divergent points