

WANDERING SUBSPACES OF THE
HARDY-SOBOLEV SPACES OVER $\mathbb{D}^{n*}$ Jiesheng XIAO (肖杰胜)[†]

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Abstract In this paper, we show that for $\frac{\log 2}{2 \log 2} \leq \beta \leq \frac{1}{2}$, suppose $\mathcal{S}$ is an invariant subspace of the Hardy-Sobolev spaces $H^2_\beta(\mathbb{D}^n)$ for the n -tuple of multiplication operators $(M_{z_1}, \dots, M_{z_n})$. If $(M_{z_1}|_{\mathcal{S}}, \dots, M_{z_n}|_{\mathcal{S}})$ is doubly commuting, then for any non-empty subset $\alpha = \{\alpha_1, \dots, \alpha_k\}$ of $\{1, \dots, n\}$, $\mathcal{W}_\alpha^{\mathcal{S}}$ is a generating wandering subspace for $M_\alpha|_{\mathcal{S}} = (M_{z_{\alpha_1}}|_{\mathcal{S}}, \dots, M_{z_{\alpha_k}}|_{\mathcal{S}})$, that is, $[\mathcal{W}_\alpha^{\mathcal{S}}]_{M_\alpha|_{\mathcal{S}}} = \mathcal{S}$, where $\mathcal{W}_\alpha^{\mathcal{S}} = \bigcap_{i=1}^k (\mathcal{S} \ominus z_{\alpha_i} \mathcal{S})$.

Key words wandering subspace; invariant subspace; Beurling's theorem; Hardy-Sobolev space; doubly commuting

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1 Introduction

A closed subspace $\mathcal{W}$ of a Hilbert space $\mathcal{H}$ is said to be a generating wandering subspace (see [4]) for a n -tuple $T = (T_1, \dots, T_n)$ of commuting bounded linear operators on $\mathcal{H}$ if

$$\mathcal{W} \perp T_1^{l_1} T_2^{l_2} \dots T_n^{l_n} \mathcal{W}$$

for all $(l_1, \dots, l_n) \in \mathbb{N}^n \setminus \{(0, \dots, 0)\}$ and

$$\mathcal{H} = \overline{\text{span}}\{T_1^{l_1} T_2^{l_2} \dots T_n^{l_n} h : h \in \mathcal{W}, l_1, \dots, l_n \in \mathbb{N}\}.$$

In this case, the tuple T is said to have the generating wandering subspace property.

Let $(T_1, \dots, T_n)$ be a commuting n -tuple of bounded linear operators on a Hilbert space $\mathcal{H}$. Does there exist a generating wandering subspace $\mathcal{W}$ for $(T_1, \dots, T_n)$?

This question has an affirmative answer for the restriction of the co-ordinate multiplication operator M_z , to an invariant subspace of the Hardy space $H^2(\mathbb{D})$. In [2], Beurling proved that if

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$\mathcal{S} \neq 0$ is an M_z -invariant subspace of the Hardy space $H^2(\mathbb{D})$, then $\mathcal{S} \ominus z\mathcal{S}$ is a one dimensional subspace spanned by an inner function η and

$$\mathcal{S} = [\eta]_{M_z|_{\mathcal{S}}} = [\mathcal{S} \ominus z\mathcal{S}]_{M_z|_{\mathcal{S}}}.$$

Beurling's theorem played an important role in operator theory, function theory and their intersection, function-theoretic operator theory. However, despite the great development in these fields over the past fifty years, it is only recently that progress was made in proving analogues for the other classical Hilbert spaces, the Dirichlet space and the Bergman space. In [8], Richter proved that the analogue of Beurling's theorem is true in the Dirichlet space $\mathcal{D}(\mathbb{D})$. It is well known that the invariant subspace lattice of the Bergman space $L_a^2(\mathbb{D})$ is very complicated. In fact, the dimension of the wandering subspace $\mathcal{S} \ominus z\mathcal{S}$ can be an arbitrary positive integer or ∞ (see [5]). However, a big breakthrough in the study of the analogue of Beurling's theorem on the Bergman space was made by Aleman, Richter and Sundberg (see [1]). They proved that any invariant subspace $\mathcal{S}$ of the Bergman space $L_a^2(\mathbb{D})$ also has the generating wandering subspace property.

In [9], Rudin showed that there are invariant subspaces $\mathcal{M}$ of $H^2(\mathbb{D}^2)$ which do not contain any bounded analytic function. In particular, for $n \geq 2$ the Beurling's theorem like characterization of $(M_{z_1}, \dots, M_{z_n})$ -invariant subspaces of $H^2(\mathbb{D}^n)$, in terms of inner functions on $\mathbb{D}^n$, is not possible. Moreover, existence of generating wandering subspaces for general invariant subspaces of the Hardy space $H^2(\mathbb{D}^n)$ rather fails spectacularly.

We now recall the following definition, we say that a closed subspace $\mathcal{M}$ of Hilbert space $\mathcal{H}$ is invariant for a n -tuple of operators $(T_1, \dots, T_n)$, if $T_i\mathcal{M} \subset \mathcal{M}$ for all $i = 1, \dots, n$. We say that a commuting n -tuple $(T_1, \dots, T_n)$ ($n \geq 2$) of bounded linear operators on $\mathcal{H}$ is said to be doubly commuting if $T_i T_j^* = T_j^* T_i$ for all $1 \leq i < j \leq n$.

Let $H(\mathbb{D}^n)$ denote the set of all holomorphic functions on $\mathbb{D}^n$, let $\mathcal{H} \subseteq H(\mathbb{D}^n)$ be a reproducing kernel Hilbert space such that the multiplication operators $\{M_{z_1}, \dots, M_{z_n}\}$ by the co-ordinate functions are bounded. Then a closed $(M_{z_1}, \dots, M_{z_n})$ -invariant subspace $\mathcal{S}$ of $\mathcal{H}$ is said to be doubly commuting if the n -tuple $(M_{z_1}|_{\mathcal{S}}, \dots, M_{z_n}|_{\mathcal{S}})$ is doubly commuting, that is, $R_i R_j^* = R_j^* R_i$ for all $1 \leq i < j \leq n$, where $R_i = M_{z_i}|_{\mathcal{S}}$.

In [11], Sarkar et al. proved that any doubly commuting invariant subspace of $H^2(\mathbb{D}^n)$ (where $n \geq 2$) has the generating wandering subspace property (see [6] for $n = 2$). Also in [7], Redett and Tung obtained the analogue conclusion for doubly commuting invariant subspaces of the Bergman space $L_a^2(\mathbb{D}^2)$. Recently, in [3], Chattopadhyay et al. proved that doubly commuting invariant subspaces of the Bergman space $L_a^2(\mathbb{D}^n)$ and the Dirichlet space $\mathcal{D}(\mathbb{D}^n)$ have the generating wandering subspace property.

Motivated by the work of [3], in this paper we will consider the generating wandering subspace property for doubly commuting invariant subspaces of the Hardy-Sobolev spaces over $\mathbb{D}^n$.

Recall that the Hardy space over the unit polydisc

$$\mathbb{D}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < 1, i = 1, \dots, n\}$$

is denoted by $H^2(\mathbb{D}^n)$ and defined by

$$H^2(\mathbb{D}^n) = \left\{ f \in H(\mathbb{D}^n) : \sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(rz)|^2 d\theta < \infty \right\},$$

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