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A SPECIAL MODULUS OF CONTINUITY AND THE K-FUNCTIONAL*

Nadezhda DOLMATOVA
Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland
E-mail: *dolmat@math.uni.wroc.pl*

Abstract We consider the questions connected with the approximation of a real continuous 1-periodic functions and give a new proof of the equivalence of the special Boman-Shapiro modulus of continuity with Peetre's K-functional. We also prove Jackson's inequality for the approximation by trigonometric polynomials.

Key words modulus of continuity; K-functional; Jackson's theorem

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1 Introduction

Denote by $C(\mathbb{T}), \mathbb{T} = \mathbb{R}/\mathbb{Z}$ the space of real continuous 1-periodic functions f with the uniform norm

$$\|f\| = \sup_{u\in\mathbb{T}} |f(u)|.$$

The derivative operator is denoted by the symbol D, and the space of functions f with $D^2 f \in C(\mathbb{T})$ will be denoted by $C^2(\mathbb{T})$.

Let $L(\mathbb{T})$ be the space of measurable, integrable functions with norm

$$\|f\|_1 = \int_{\mathbb{T}} |f(u)|\mathrm{d}u$$

and let T_{n-1} be the set of real trigonometric 1-periodic polynomials τ of degree at most $n-1$:

$$\tau(t) := \sum_{j=-n+1}^{n-1} \alpha_j \exp(2\pi i j t), \qquad \alpha_j = \overline{\alpha}_{-j}.$$

For $f \in C(\mathbb{T})$, we denote by $E_{n-1}(f)$ the value of the best approximation of f by real trigonometric polynomials of degree at most $n-1$,

$$E_{n-1}(f) := \inf_{\tau\in T_{n-1}} \|f - \tau\|.$$

We will use the convolution of periodic functions f with positive functions g, with finite support. In this case, the convolution can be understood in the following sense:

$$(f * g)(t) := \int_{\mathbb{R}} f(u)\, g(t-u)\mathrm{d}u.$$

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We denote by $\chi_h^k,\ k=1,2,\cdots$ the convolution powers of the normalized characteristic function of the interval $(-h/2,h/2),\ h>0$:

$$\chi_h^k := \chi_h^{k-1} * \chi_h, \quad \chi_h(t) := \begin{cases} \dfrac{1}{h} & t\in(-h/2,h/2), \\ 0, & t\notin(-h/2,h/2). \end{cases}$$

In particular,

$$\chi_h^2(t) = \begin{cases} \dfrac{1}{h}\left(1-\dfrac{|t|}{h}\right), & t\in(-h,h), \\ 0, & t\notin(-h,h). \end{cases}$$

The functions χ_h^k are the cardinal B-splines with support $[-kh/2,kh/2]$ and $\|\chi_h^k\|_1=1$.

We will use the following moduli of continuity (see [1, 2, 9])

$$W_2(f,\chi_h^k) := \|f - f*\chi_h^k\|,$$
$$W_2^*(f,\chi_h^k) := \sup_{0<u\le h} W_2(f,\chi_u^k).$$

They are special cases of the Boman-Shapiro moduli of continuity (see [3, 4, 11]).

This paper is the continuation of [1]. The main result of [1] is the following Jackson inequality for the uniform approximation of continuous 1-periodic functions by trigonometric polynomials.

Let f be a continuous 1-periodic function and $n\in\mathbb{N},\ h=\alpha/(2n),\ \alpha>2/\pi$. Then the following inequality holds

$$E_{n-1}(f) \le (\sec 1/\alpha + \tan 1/\alpha)\, W_2(f,\chi_h). \tag{1.1}$$

The estimate is exact for $\alpha=1,3,\cdots$.

In [1] the following sharp Bernstein-Nikolsky-Stechkin inequality for $\tau\in T_n$ was also obtained.

Let τ be a real trigonometric 1-periodic polynomial of degree at most $n-1$ for $n\in\mathbb{N}$, and suppose $h\in(0,1/n]$. Then

$$\|D^2\tau\| \le (2\pi n)^2\ W_2(c_n,\chi_h)^{-1}\, W_2(\tau,\chi_h), \quad c_n(t) := \cos(2\pi nt). \tag{1.2}$$

The Jackson inequality (1.1) and the Bernstein-Nikolsky-Stechkin estimate (1.2) allowed us to prove the equivalence of a special modulus of continuity and the second Peetre's K-functional [1].

Let $h\in(0,1]$. Then

$$1/4K_2(f,h/(4\sqrt{6})) \le W_2(f,\chi_h) \le 4K_2(f,h/(4\sqrt{6})). \tag{1.3}$$

The equivalence of moduli of this type and the K-functional is known (see [7] and [14]). Here we give a new form of this equivalence with the calculation of the constants. We present a new simple proof of the estimates of the type (1.3) with better constants (Theorem 3.1). Theorem 3.1 and a new construction in the proof of Theorem 3.1 are the main results of the present paper. Further, we introduce a generalized K-functional which is related to the new approach to the direct theorems of approximation theory [1, 9] and give the analogue of Theorem 3.1 for it (Theorem 3.2). We also give a proof of the estimates of type (1.1) which hold for $\alpha>0$ and better than (1.1) for $\alpha<0.778$ (Theorem 4.1).