Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2015,**35B**(1):140–156
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

GLOBAL CLASSICAL SOLUTIONS FOR QUANTUM KINETIC FOKKER-PLANCK EQUATIONS*

Lan LUO (罗兰)
School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China; Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institutes, Guangzhou University, Guangzhou 510006, China
E-mail: *lanluogu@sina.com*

Xinping ZHANG (张新平)
Department of Mathematics and Sciences, Luoyang Institute of Science and Technology, Luoyang 471023, China
E-mail: *zhangxply@sina.com*

Abstract We consider a class of nonlinear kinetic Fokker-Planck equations modeling quantum particles which obey the Bose-Einstein and Fermi-Dirac statistics, respectively. We establish the existence and convergence rate to the steady state of global classical solution to such kind of equations around the steady state.

Key words quantum Fokker-Planck equations; energy method; convergence rates

2010 MR Subject Classification 35Q99; 35B40

1 Introduction

In this paper we are interested in the following quantum kinetic Fokker-Planck equation:

$$\partial_t f + p \cdot \nabla_x (f + \sigma\kappa f^2) = \nabla_p \cdot (pf(1+\kappa f) + \nabla_p f) \tag{1.1}$$

with initial data $f(0,x,p) = f_0(x,p)$. Here $\sigma = 1$, or $\sigma = 0$ and $f(t,x,p)$ denotes the particles distribution function on phase space $\mathbf{R}_x^3 \times \mathbf{R}_p^3$ for any time $t \geq 0$.

Equation (1.1) with $\sigma = 1$ was introduced in [18], for the classical particles obeying an exclusion principle. A formal derivation from the generalized quantum Boltzmann equation and the Uehling-Uhlenbeck equation was given in [16, 26]. Different physical applications can be founded in [13, 17] and the references therein. Notice that $\kappa = -1$ corresponds to the Fermions and $\kappa = 1$ to the Bosons. For $\kappa = 0$, equation (1.1) simplifies to the classical linear Fokker-Planck equation. Equation (1.1) with $\sigma = 0$ was proposed in [27, 28] to describe self-gravitating particles and the formation of Bose-Einstein condensates in a kinetic framework.

*Received December 13, 2013; revised March 10, 2014. This research was supported by the National Natural Science Foundation of China (11371151).

The purpose of this paper is to construct global classical solutions and large time behavior of global solution to equation (1.1) near an steady state, which is given by

$$f_\infty(p) = \frac{1}{\exp\left(\frac{|p|^2}{2} + \theta\right) - \kappa}. \tag{1.2}$$

Distribution (1.2) is the well-known Fermi-Dirac equilibrium distribution for $\kappa = -1$. For $\kappa = 1$, $f_\infty(p)$ is the so-called regular Bose-Einstein distribution. If $\kappa = 0$, $f_\infty(p)$ simplifies to the classical Maxwellian. In this paper we require $\theta > 0$ in (1.2) and we see [5, 11, 25] for a detailed discussion.

We define the perturbation $g(t, x, p)$ around this steady state by

$$f = f_\infty + \sqrt{\mu_\infty} g, \tag{1.3}$$

where $\mu_\infty = f_\infty + \kappa f_\infty^2$. Then equation (1.1) yields the following equation for the perturbation $g(t, x, p)$:

$$\partial_t g + (1 + 2\sigma\kappa f_\infty) p \cdot \nabla_x g + Lg = Q(g) \tag{1.4}$$

with initial data $g(0, x, p) = g_0(x, p)$. Here the linearized collision operator L is given by

$$\begin{aligned} Lg &= -\frac{1}{\sqrt{\mu_\infty}} \nabla_p \cdot \left(\nabla_p (g\sqrt{\mu_\infty}) + p\eta_\infty g\sqrt{\mu_\infty}\right) \\ &= -\Delta_p g - g\Big(\frac{3}{2}\eta_\infty - |p|^2\Big(\frac{1}{4} + 2\kappa\mu_\infty\Big)\Big), \end{aligned} \tag{1.5}$$

where $\eta_\infty = 1 + 2\kappa f_\infty$. The quadratic remainder $Q(g)$ is

$$Q(g) = \frac{\kappa}{\sqrt{\mu_\infty}} \Big(\nabla_p \cdot (p\mu_\infty g^2) - \sigma\mu_\infty p \cdot \nabla_x (g^2)\Big). \tag{1.6}$$

In this paper, the following notations are used. Let α and β be $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$, respectively. Denote

$$\partial_\beta^\alpha \equiv \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{p_1}^{\beta_1} \partial_{p_2}^{\beta_2} \partial_{p_3}^{\beta_3}.$$

If each component of β is not greater than corresponding one of $\overline{\beta}$, we use the standard notation $\beta \le \overline{\beta}$. And $\beta < \overline{\beta}$ means that $\beta \le \overline{\beta}$ and $|\beta| < |\overline{\beta}|$. $C_\beta^{\overline{\beta}}$ is the usual binomial coefficient. For the study of the time decay rate, the space $Z_q = L^2(\mathbf{R}_p^3; L^q(\mathbf{R}_x^3))$ is used with its norm defined by

$$\|f\|_{Z_q} = \left(\int_{\mathbf{R}^3} \left(\int_{\mathbf{R}^3} |f(x,p)|^q \mathrm{d}x\right)^{\frac{2}{q}} \mathrm{d}p\right)^{\frac{1}{2}}.$$

We will use $\langle \cdot, \cdot \rangle$ to denote the standard L^2 inner product in $\mathbf{R}_p^3$, and $(\cdot, \cdot)$ for the one in $\mathbf{R}_x^3 \times \mathbf{R}_p^3$. $|\cdot|_2$ denotes the L^2 norm in $\mathbf{R}_p^3$ and $\|\cdot\|$ to denote the L^2 norms in $\mathbf{R}_x^3 \times \mathbf{R}_p^3$. From now on, C or c denotes a generic positive constant which may vary from line to line.

First note that L is self-adjoint on $L^2(\mathbf{R}_p^3)$ and by a simple calculation one has

$$\langle Lg, g \rangle = \int_{\mathbf{R}^3} \left| \nabla_p g + \frac{p}{2}\eta_\infty g \right|^2 \mathrm{d}p = \int_{\mathbf{R}^3} \left| \nabla_p \left(\frac{g}{\sqrt{\mu_\infty}}\right) \right|^2 \mu_\infty \mathrm{d}p. \tag{1.7}$$

Thus the kernel of positive operator L is given by

$$\mathcal{N} = \mathrm{span}\{\sqrt{\mu_\infty}\}. \tag{1.8}$$