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# ON THE EXISTENCE OF LOCAL CLASSICAL SOLUTION FOR A CLASS OF ONE-DIMENSIONAL COMPRESSIBLE NON-NEWTONIAN FLUIDS*

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**Abstract** In this paper, the aim is to establish the local existence of classical solutions for a class of compressible non-Newtonian fluids with vacuum in one-dimensional bounded intervals, under the assumption that the data satisfies a natural compatibility condition. For the results, the initial density does not need to be bounded below away from zero.

**Key words** compressible non-Newtonian fluids; vacuum; local classical solution

**2010 MR Subject Classification** 76N10; 35M10; 35Q30

## 1 Introduction

In this paper, we consider the classical solutions to the initial boundary value problems for a class of compressible non-Newtonian fluids with vacuum in one-dimensional bounded interval:

$$\begin{cases} \rho_t + (\rho u)_x = 0, & (x,t) \in [0,1] \times (0,T), \\ (\rho u)_t + (\rho u^2)_x - ((u_x^2 + \mu_0)^{\frac{p-2}{2}} u_x)_x + \pi_x = 0, & (x,t) \in [0,1] \times (0,T), \\ \pi \equiv \pi(\rho) = A\rho^\gamma, & A > 0, \ \gamma > 1. \end{cases} \tag{1.1}$$

The system comes from non-Newtonian fluid mechanics that $\rho$, $u$ and $\pi$ denote the fluid density, velocity and pressure, respectively. Here, $\gamma$ is the adiabatic gas exponent. The constant $1 < p < 2$ and $\mu_0 > 0$ are given. Equations (1.1) is studied with the initial and boundary conditions:

$$\begin{cases} (\rho, u)|_{t=0} = (\rho_0, u_0), & x \in [0,1], \\ u|_{x=0} = u|_{x=1} = 0, & t \in [0,T]. \end{cases} \tag{1.2}$$

In recent years, considerable attention was paid to the mathematical model of non-Newtonian fluids, which are described by a set of equations including a tress tensor depending in a non-linear way by the gradient of the velocity. One of the first mathematical investigations of such models was carried out by Ladyzhenskaya (see [16]), she considered the following system of equations there known as model of incompressible non-Newtonian fluid

$$\begin{cases} u_t - \mathrm{div} a(Du) + \nabla \pi = -\mathrm{div}(u \otimes u) + f, \\ \mathrm{div} u = 0, \end{cases}$$

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where $Du$ denotes the symmetric part of the gradient $\nabla u$ and $\pi$ denotes the pressure. The main point in the previous system is that the monotone vector field $a$ depends in a non-linear way by $Du$

$$a(Du) = ((\mu_0 + \mu_1|Du|^2)^{\frac{p-2}{2}})Du,$$

where $p > 1$. In last decades, mathematical aspects for the incompressible non-Newtonian fluids have been extensively studied and significant progress has been made (see [1, 11, 27–31, 33]). In particular, Màlek and his collaborators did several results (see [22]) on the weak solutions.

It is natural to ask what about the results on solutions to the compressible non-Newtonian fluid. Noting that $p = 2$, we obtain the Navier-stokes equations. Lions in [20] used the weak convergence method and first showed the existence of global weak solutions for an isentropic fluid under the assumption that $\gamma \geqslant \frac{3}{2}$ if $N = 2$ and $\gamma > \frac{9}{5}$ if $N = 3$. Feireisl in [9] extended Lions' global existence result in $\mathbb{R}^3$ to the case $\gamma > \frac{3}{2}$. On the classical solutions, Cho and Kim in [4] in 2006 got the local existence and uniqueness result for $N = 3$ by using successive approximations, based on some a priori estimates for the solutions to the corresponding linearized problems. Ding, Wen and Zhu in [6] in 2011 got the local existence and uniqueness result for 1D compressible isentropic Navier-Stokes equations with large initial data, density-dependent viscosity, external force and vacuum. For more results about the existence, uniqueness, or other virtues of the solutions of the Navier-Stokes equation, we may refer to [2, 5, 13, 14, 21, 26, 32, 34, 35].

For compressible non-Newtonian models, Mamontov proved the global existence of weak solution for multi-dimensional case (see [23–25]). Yuan and Xu [38] established an existence result on the local strong solutions with nonnegative densities in the case $1 < p < 2$. In [7], Fang and Guo proved the blowup criterion for the local strong solutions obtained in [38]. For more related results, we refer the reader to [8, 37, 39, 40] and the references therein.

In this paper, we concentrate on the classical solutions for the initial boundary value problem (1.1)–(1.2) with $1 < p < 2$ and $\mu_0 > 0$. Our main concern here is to show the local existence of classical solutions to (1.1)–(1.2) with nonnegative initial densities. We prove the existence of solution in $C([0, T_*]; H^2 \times (H^3 \cap H_0^1))$ under certain compatibility condition on the initial data.

Notation used throughout this paper $I = [0, 1]$, $\partial I = \{0, 1\}$, $\Omega_T = I \times [0, T]$ for $T > 0$. For $r \geqslant 1$, $L^r = L^r(I)$ denotes the $L^r$ space with the norm $|\cdot|_{L^r}$. For $k \geqslant 1$ and $r \geqslant 1$, $W^{k,r} = W^{k,r}(I)$ denotes the Sobolev space whose norm is denoted as $|\cdot|_{k,r}$ and $H^k = W^{k,2}$. For an integer $k \geqslant 0$ and $0 < \alpha < 1$, let $C^{k+\alpha}(I)$ denote the Schauder space of functions on $I$, whose $k$-th order derivative is Hölder continuous with exponents $\alpha$, with the norm $|\cdot|_{C^{k+\alpha}}$.

Our main result in this paper is in the following.

**Theorem 1.1** Assume that $\rho_0 \geqslant 0$, $\rho_0^\gamma \in H^2$, $u_0 \in H_0^1 \cap H^3$. Assume further that the initial date $\rho_0$, $u_0$ satisfy the compatibility condition

$$-((u_{0x}^2 + \mu_0)^{\frac{p-2}{2}} u_{0x})_x + \pi_x(\rho_0) = \rho_0 g \text{ for some } g \in H_0^1. \tag{1.3}$$

Then there exist a small time $T_* \in (0, +\infty)$ and a unique local classical solution $(\rho, u)$ to (1.1)–(1.2) satisfying

$$\begin{aligned}
&\rho \geqslant 0,\ (\rho, \rho^\gamma) \in C([0, T_*]; H^2),\ (\rho_t, (\rho^\gamma)_t) \in C([0, T_*]; H^1),\\
&\rho_{tt} \in C([0, T_*]; L^2),\ (\rho^\gamma)_{tt} \in C([0, T_*]; L^2),\\
&u \in C([0, T_*]; H_0^1 \cap H^3),\ u_t \in C([0, T_*]; H^2),
\end{aligned}$$