Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2015,**35B**(1):216–234
Mathematica Acta Scientia
数学物理学报
http://actams.wipm.ac.cn

TIME PERIODIC SOLUTIONS OF NON-ISENTROPIC COMPRESSIBLE MAGNETOHYDRODYNAMIC SYSTEM*

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Abstract In this paper, we study the non-isentropic compressible magnetohydrodynamic system with a time periodic external force in $\mathbb{R}^n$. Under the condition that the optimal time decay rates are obtained by spectral analysis, we show that the existence, uniqueness and time-asymptotic stability of time periodic solutions when the space dimension $n \geq 5$. Our proof is based on a combination of the energy method and the contraction mapping theorem.

Key words non-isentropic compressible magnetohydrodynamic system; time periodic solution; optimal time decay rates; energy estimates

2010 MR Subject Classification 35M10; 35Q35; 35B10

1 Introduction

The motion of electrically conducting media in the presence of a magnetic can be described by the non-isentropic compressible magnetohydrodynamic system (MHD):

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho, \theta) - (\nabla \times H) \times H = \nabla \cdot \psi + \rho f(t, x), \\ \varepsilon_t + \nabla \cdot (u(\rho e + \frac{1}{2}\rho u^2 + P)) + \nabla \cdot (\nu(\nabla \times H) \times H - (u \times H) \times H) \\ \qquad = \nabla \cdot (u\psi) + \nabla \cdot (\kappa \nabla \theta) + \rho f \cdot u, \\ H_t - \nabla \times (u \times H) = \nu \Delta H, \qquad \nabla \cdot H = 0. \end{cases} \tag{1.1}$$

Here $\rho > 0$, $u = (u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$, $P = P(\rho, \theta)$ and $H = (H_1, H_2, \cdots, H_n) \in \mathbb{R}^n$ represent the density, the velocity, the pressure and magnetic field respectively. Furthermore, the viscous stress tensor ψ is formulated by $\psi = \mu(\nabla u + \nabla u^T) + \lambda \nabla \cdot u I$ with I the identity matrix, the total energy is given by $\varepsilon = \rho(e + \frac{1}{2}|u|^2) + \frac{1}{2}|H|^2$ with $e = C_v\theta$, and the positive constants μ, λ are the viscosity coefficients with $\lambda + \frac{2}{n}\mu \geq 0$, the constants $C_v > 0, \kappa > 0, \nu > 0$ are the heat capacity at constant volume, the coefficient of heat conductivity and the magnetic diffusivity acting as a magnetic diffusion respectively. In addition, $f(t, x)$ is a given external force.

*Received November 26, 2013; revised March 17, 2014. Supported by National Natural Science Foundation of China (11271305).

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In this paper, we study the time periodic solution to problem (1.1) for (ρ, u, θ, H) around a constant state $(\rho_\infty, 0, \theta_\infty, 0)$ for $n \geq 5$, where ρ_∞ and θ_∞ are positive constants. Meanwhile $P = P(\rho, \theta)$ is smooth in a neighborhood of $(\rho_\infty, \theta_\infty)$ satisfying $P_\rho(\rho_\infty, \theta_\infty) > 0$ and $P_\theta(\rho_\infty, \theta_\infty) > 0$, and f is time periodic with period $T > 0$.

The main goal of the present paper is to show that problem (1.1) admits a time periodic solution around the constant state $(\rho_\infty, 0, \theta_\infty, 0)$ for $n \geq 5$ which has the same period as f. With the energy method and the optimal decay estimates of solutions to the linearized system, we are able to derive the existence and uniqueness of time periodic solution in some suitable function space by the contraction mapping theorem.

Precisely, let $N \geq n + 2$ be a positive integer, define the solution space by

$$X_d(0,T) = \left\{ (\rho, u, \theta, H)(t,x) \left| \begin{array}{c} \rho(t,x) \in C(0,T; H^{N-1}(\mathbb{R}^n)) \cap C^1(0,T; H^{N-2}(\mathbb{R}^n)), \\ (u,\theta,H)(t,x) \in C(0,T; H^{N-1}(\mathbb{R}^n)) \cap C^1(0,T; H^{N-3}(\mathbb{R}^n)), \\ \nabla\rho(t,x) \in L^2(0,T; H^{N-1}(\mathbb{R}^n)), \\ \nabla(u,\theta,H)(t,x) \in L^2(0,T; H^{N}(\mathbb{R}^n)), |||(\rho,u,\theta,H)||| \leq d \end{array} \right. \right\}$$

for some positive constant d and with the norm $||| \cdot |||$ given by

$$|||(\rho,u,\theta,H)|||^2 = \sup_{0 \leq t \leq T} \|(\rho,u,\theta,H)(t)\|_{N-1}^2 + \int_0^T \left(\|\nabla\rho(t)\|_{N-1}^2 + \|\nabla(u,\theta,H)(t)\|_N^2 \right) \mathrm{d}t.$$

There are some studies on the time periodic solution, such as the compressible Navier-Stokes equations, the Boltzmann equation, the compressible Navier-Stokes-Korteweg system and isentropic magnetohydrodynamic equations, cf. [4, 6, 18, 20, 22, 24]. It is worth mentioning the space dimension where our results hold. In Section 4, L^2 optimal decay rate of the solutions to the linear homogeneous system of (2.4) is established as

$$\|(\varrho, v, \vartheta, B)(t)\| \leq C(1+t)^{-\frac{n}{4}} \|(\varrho_0, v_0, \vartheta_0, B_0)\|_{H^l \cap L^1},$$

$$\|\nabla(\varrho, v, \vartheta, B)(t)\|_{l-1} \leq C(1+t)^{-\frac{n}{4}-\frac{l}{2}} \|(\varrho_0, v_0, \vartheta_0, B_0)\|_{H^l \cap L^1}$$

for the initial data $(\varrho_0, v_0, \vartheta_0, B_0) \in H^l \cap L^1$, so when, and only when $n \geq 5$, we can show the integral in (5.4) is convergent. Thus, similar to the case of compressible Navier-Stokes equations, Theorem 1.1 is obtained only in the case $n \geq 5$ because of the convergence of the integral in (5.4). However, how to deal with the case when $n < 5$, and especially the physical case when $n = 3$, remains an open problem.

There are a lot of works on the existence, stability and convergence rates of smooth solutions, strong solutions and weak solutions to the isentropic or non-isentropic compressible MHD system, in virtue of the physical importance, complexity and wide range of applications for the MHD system. In particular, the one-dimensional problem was studied in many papers, we can refer to [2, 3, 9, 13, 26] and the references therein. Kawashima [15] proved the global existence of smooth solutions to the general electromagnetic fluid equations in the two-dimensional case with the initial data a small perturbation of some given constant state. Umeda et al. [23] studied the global existence and the time decay of smooth solutions to the linearized MHD equations. The local existence of strong solutions to the nonlinear compressible MHD equations was proven by Volpert and Khudiaev [25] and Fan and Yu [8]. Hu and Wang [10, 11] and Fan and Yu [7] considered the global weak solutions to the nonlinear compressible MHD equations