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SOLUTIONS TO NONLINEAR ELLIPTIC EQUATIONS WITH A GRADIENT*

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Abstract In this article, we consider existence and nonexistence of solutions to problem

$$\begin{cases} -\Delta_p u + g(x,u)|\nabla u|^p = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

with $1 < p < \infty$, where f is a positive measurable function which is bounded away from 0 in Ω, and the domain Ω is a smooth bounded open set in $\mathbb{R}^N (N \geq 2)$. Especially, under the condition that $g(x,s) = 1/|s|^\alpha$ $(\alpha > 0)$ is singular at $s = 0$, we obtain that $\alpha < p$ is necessary and sufficient for the existence of solutions in $W_0^{1,p}(\Omega)$ to problem (0.1) when f is sufficiently regular.

Key words quasilinear elliptic equations; existence and nonexistence; gradient terms; singular weights

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1 Introduction

In this work, we study existence and nonexistence of solutions to the problem

$$\begin{cases} -\Delta_p u + g(x,u)|\nabla u|^p = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, and the operator Δ_p stands for the p-Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Throughout this paper, we assume that for any compact subset $\omega \subset \Omega$, there exists a positive constant f_ω such that

$$f \geq f_\omega \quad \text{in } \omega. \tag{1.2}$$

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For the problem

$$\begin{cases} -\Delta_p u + g(x,u)|\nabla u|^q = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 \le q \le p$, $g(x,u) \in L^1(\mathbb{R})$, and the data f in suitable Lebesgue spaces, has been exhaustively studied. We refer the readers to [1–4] and the references therein for semilinear elliptic problem, and to [5–15] and the references therein for quasilinear elliptic problem. If $g(x,u) = 1/|u|^\alpha$ with $\alpha > 0$ is singular at $u = 0$, the existence of solutions to (1.1) with $p = 2$ has been obtained in [16–18] for $0 < \alpha \le 1$, and the authors in [19] have proved that $\alpha < 2$ is necessary and sufficient for the existence of solutions in $H_0^1(\Omega)$ for every f with sufficient regularity and satisfying (1.2).

In this article, we extend the results of [19] to general case $p > 1$. That is, we consider that $g(x,u)$ is singular at $u = 0$ for a.e. $x \in \Omega$, and obtain existence and nonexistence of solutions to problem (1.1). Here is our main result.

Theorem 1.1 Suppose $p > 1$, $f \in L^1(\Omega)$ such that (1.2) holds. Let $h : (0, +\infty) \to [0, +\infty)$ be a continuous nonnegative function which is nonincreasing in a neighborhood of 0, and assume that

$$\lim_{s \to 0^+} \int_s^1 h(t)^{1/p} \mathrm{d}t < +\infty. \tag{1.3}$$

If

$$0 \le g(x,s) \le h(s) \quad \text{for a.e. } x \in \Omega \text{ and every } s > 0, \tag{1.4}$$

then problem (1.1) admits a positive solution in $W_0^{1,\sigma}(\Omega)$ for any $\max\{1, p-1\} \le \sigma < \frac{N(p-1)}{N-1}$.

Furthermore, if there exist constants $s_0 > 0$ and $g_0 > 0$ such that

$$g(x,s) \ge g_0, \quad \forall x \in \Omega, \ \forall s \ge s_0, \tag{1.5}$$

then the solution belongs to $W^{1,p}(\Omega)$.

2 Preliminaries

Let $\bar{p} = \max\{1, p-1\}$, we call $u \in W_0^{1,\bar{p}}(\Omega)$ is a positive solution of problem (1.1), means that $u > 0$ in Ω, $g(x,u)|\nabla u|^p \in L^1(\Omega)$, and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \psi \mathrm{d}x + \int_\Omega g(x,u)|\nabla u|^p \psi \mathrm{d}x = \int_\Omega f \psi \mathrm{d}x, \quad \forall \psi \in C_c^1(\Omega).$$

If $u \in W_0^{1,p}(\Omega)$ is a solution of problem (1.1), then we have

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \psi \mathrm{d}x + \int_\Omega g(x,u)|\nabla u|^p \psi \mathrm{d}x = \int_\Omega f \psi \mathrm{d}x, \quad \forall \psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

To obtain the existence of solutions to problem (1.1), we need to study the behavior of $g(x,u)$ at the neighborhood of $u = 0$. The following result shows that the solution u is bounded below away from 0 in every compact subset of Ω, and hence, we conclude that $g(x,u)$ is bounded in every compact subset contained in Ω, if u is bounded.