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# SUB-ADDITIVE PRESSURE ON A BOREL SET*

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**Abstract** The goal of this paper is to investigate topological conditional pressure of a continuous transformation as defined for sub-additive potentials. This study presents a variational inequality for sub-additive topological conditional pressure on a closed subset, which is the other form of the variational principle for the sub-additive topological pressure presented by Cao, Feng, and Huang in [9]. Moreover, under some additional assumptions, this result can be generalized to the non-compact case.

**Key words** sub-additive potentials; topological pressure; conditional entropy; variational inequality

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## 1 Introduction

In this study, $(X, T)$ denotes a topological dynamical system (TDS for short) in the sense that $T : X \to X$ is a continuous map on a compact metric space $X$ with the metric $d$.

Entropy is a critical factor in dynamical systems and the study of ergodic theory. Metric entropy was defined by Kolomogorov and Sinai from Shannon's information theory in 1959, while Adler, Konheim and McAndrew [1] used the concept of open covers to introduce topological entropy in 1965, and Bowen [7] later defined topological entropy on a metric space by using generating and separating sets, respectively. These three different definitions of topological entropy was proved equivalent provided the compactness of the space, see [23] for the details. Metric or measure-theoretic entropy measures the maximal loss of information in the iteration of finite partitions in a measure-preserving transformation, whereas topological entropy measures the maximal exponential growth rate of orbits. However, these concepts are not isolated. These two notions are connected by a famous variational principle stating that the topological

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entropy is the supremum of the metric entropies for all invariant probability measures of a given topological system and is described in the following:

$$h(T) = \sup\{h_\mu(T) : \mu \in M(X,T)\},$$

where $h(T)$ denotes the topological entropy for $T$ and $h_\mu(T)$ is the metric entropy. The term $M(X,T)$ denotes all the $T$-invariant Borel probability measures on $X$.

As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in the thermodynamic formalism. Topological pressure contains information about the dynamics of the system, which can be extracted by varying the potential energy function. It is well known that the topological pressure with a potential function plays a fundamental role in the study of the Hausdorff dimension of repellers and the hyperbolic set. Related studies include [4, 7, 13, 14, 17, 19, 21–23]. The relationship among topological pressure, potential function and metric entropy are formulated by a variational principle, which states that if $T : X \to X$ is a continuous map, $f : X \to \mathbb{R}$ is a continuous function, and $P(T,f)$ denotes the topological pressure of $T$ with respect to $f$, then

$$P(T,f) = \sup\Big\{h_\mu(T) + \int f\mathrm{d}\mu : \mu \in M(X,T)\Big\}.$$

In [11], Falconer considered the topological pressure for sub-additive potentials, and proved the variational principle for sub-additive topological pressure under some Lipschitz conditions and bounded distortion assumptions on the sub-additive potentials. For an arbitrary subset and any non-additive potentials on the compact metric space, Barreira [3] defined non-additive topological pressure and also showed the variational principle under particular convergence conditions on the potentials. Recently, Cao, Feng and Huang [9] generalized the results of Rulle and Walters to the sub-additive potentials on the compact metric space. It deserves to mention that they didn't put any other assumptions on the potentials and their results had relevant applications in dimension theory, see [2]. Feng and Huang in [12] introduced the notion of asymptotically additive/subadditive potentials, and proved the variational principle for topological pressure with these two potential. The main goal of demonstration is to study the multifractal analysis of these two potentials. Note that Barreira [5, 6] and Mummert [18] dealt with variational principle for topological pressure with almost additive potentials. Huang, Yi [16] and Zhang [24] considered the variational principle for the local topological pressure. For more information, see [25] and [26] for variational principle of conditional topological pressure and coset pressure with sub-additive potentials, respectively.

Furthermore, Li, Chen and Cheng [15] generalized the results of Walters regarding the variational principle for topological pressure and their results can be stated precisely as follows. Let $T : X \to X$ be a continuous map, and $f : X \to \mathbb{R}$ be a continuous function. Given a $T$-invariant closed subset $G$ of $X$, i.e., $T^{-1}G = G$, and consider the pressure $P_G(T,f)$ of $T$ on $G$ with respect to $f$, then the following variational inequality is obtained

$$P_G(T,f) \le \sup_{\mu\in M(X,T)} \Big\{h_\mu(T|\langle G\rangle) + \int f\mathrm{d}\mu\Big\} \le \max\Big\{P_G(T,f), P_{\overline{X\setminus G}}(T,f)\Big\}, \tag{1.1}$$

where $\langle G\rangle$ denotes the partition $\{G, X\setminus G\}$, and $h_\mu(T|\langle G\rangle)$ denotes the conditional entropy of $T$ with respect to $\mu$, see [23] for the details. This formula can be used to generalize the results of Cheng [10] regarding topological entropy. Inspired by [15] and [9], this paper considers the