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NORMAL FAMILIES OF MEROMORPHIC FUNCTIONS WITH SHARED VALUES*

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Abstract We obtain some normality criteria of families of meromorphic functions sharing values related to Hayman conjecture, which improves some earlier related results.

Key words meromorphic function; shared value; normal family

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1 Introduction and Main Results

Let $\mathbb{C}$ be the set of complex numbers and let D be a domain in $\mathbb{C}$, which means that D is a connected non-empty open subset of $\mathbb{C}$. Let $\mathcal{M}(D)$ be the set of meromorphic functions defined in D. For $\{f,g\}\subset\mathcal{M}(D)$, $\{a,b\}\subset\mathbb{P}^1=\mathbb{C}\cup\{\infty\}$, we will write $f=a\Rightarrow g=b$ (resp., $f=a\Leftrightarrow g=b$) if $f^{-1}(a)\subset g^{-1}(b)$ (resp., $f^{-1}(a)=g^{-1}(b)$), and say that f and g share a IM (ignoring multiplicities) if $f^{-1}(a)=g^{-1}(a)$ (see [12]).

Let $\mathcal{F}$ be a family in $\mathcal{M}(D)$. Montel first obtained a famous normal criterion which states that $\mathcal{F}$ is normal in D if each element of $\mathcal{F}$ omits three distinct values a,b,c in $\mathbb{P}^1$, where the family $\mathcal{F}$ is said to be normal in D if any sequence of $\mathcal{F}$ must contain a subsequence which locally uniformly spherically converges to a meromorphic function or ∞ in D (see [7, 12]).

Later, Carathéodory [2] relaxed the fixed values a,b,c in Montel's theorem to "wandering" constants a_f,b_f,c_f depending on $f\in\mathcal{F}$ such that their spherical distances have a uniform lower bound ε, that is,

(i) $\min\{\sigma(a_f,b_f),\sigma(a_f,c_f),\sigma(b_f,c_f)\}\geq\varepsilon$ for some positive real number ε.

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The definition of the spherical distance $\sigma(x,y)$ between $x\in\mathbb{P}^1$ and $y\in\mathbb{P}^1$ can be found in [1].

According to the idea of Carathéodory, Singh and Singh [13] generated a result of Pang and Zalcman [11] by proving that $\mathcal{F}$ is normal in D if each $f\in\mathcal{F}$ satisfies $f=0\Leftrightarrow f'=0$ and $f=c_f\Leftrightarrow f'=b_f$, where c_f, b_f are nonzero complex numbers such that spherical distances of $0, b_f, c_f$ have a uniform lower bound ε, that is,

(i)' $\min\{\sigma(0,b_f),\sigma(0,c_f),\sigma(b_f,c_f)\}\geq\varepsilon$ for some positive real number ε,

and such that

(ii) $\frac{b_f}{c_f}$ is independent of f.

Moreover, Grahl [6] corrected a gap of the proof of another similar result in the paper of Singh and Singh [13].

We improve the normality criteria related to normality conjecture due to Hayman [7] which was proved by several authors (see [3, 5, 9]) based on the ideas of shared values as follows:

Theorem 1.1 Let n be a positive integer and let $\mathcal{F}$ be a family in $\mathcal{M}(D)$ such that each element f in $\mathcal{F}$ satisfies $f^nf'=\alpha_f\Leftrightarrow f'=b_f$ for nonzero complex numbers α_f, b_f. If there exists $c_f\in\mathbb{C}$ satisfying $c_f^{n+1}=\alpha_f$ for each $f\in\mathcal{F}$ such that (i)' and (ii) hold, then $\mathcal{F}$ is normal in D.

Obviously, Theorem 1.1 implies directly the following fact:

Corollary 1.2 Let n be a positive integer and take nonzero complex numbers α, b with $\alpha^{\frac{1}{n+1}}\neq b$. Then a family $\mathcal{F}$ in $\mathcal{M}(D)$ is normal in D if each $f\in\mathcal{F}$ satisfies $f^nf'=\alpha\Leftrightarrow f'=b$.

By weakening the conditions in Theorem 1.1, we can obtain the following result:

Theorem 1.3 Let n be a positive integer and let $\mathcal{F}$ be a family in $\mathcal{M}(D)$ such that each element f in $\mathcal{F}$ satisfies

$$f^nf'=\alpha_f\Rightarrow f'=b_f,\, f'=a_f\Rightarrow f=\beta_f$$

for nonzero complex numbers $a_f, b_f, \alpha_f, \beta_f$. If there exists $c_f\in\mathbb{C}$ satisfying $c_f^{n+1}=\alpha_f$ for each $f\in\mathcal{F}$ such that (i) holds and such that

(ii)' $\frac{a_f}{c_f}$, $\frac{b_f}{c_f}$, $\frac{\beta_f}{c_f}$ are independent of f respectively,

then $\mathcal{F}$ is normal in D.

Corollary 1.4 Let n be a positive integer and take four nonzero complex numbers a, b, α, β such that a, b, c are distinct with $c^{n+1}=\alpha$. Then a family $\mathcal{F}$ in $\mathcal{M}(D)$ is normal in D if each $f\in\mathcal{F}$ satisfies $f^nf'=\alpha\Rightarrow f'=b$ and $f'=a\Rightarrow f=\beta$.

2 Some Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1 (see [10, 14]) Let k be a positive integer and let $\mathcal{F}$ be a family in $\mathcal{M}(D)$ such that each $f\in\mathcal{F}$ has only zeros of multiplicity at least k. If $\mathcal{F}$ is not normal at $z_0\in D$, then for each real number α with $-1<\alpha<k$, there exist a sequence $\{z_n\}$ in D converging to z_0, a sequence $\{\rho_n\}$ of positive numbers converging to 0, and a sequence $\{f_n\}$ in $\mathcal{F}$ such that functions $g_n(\zeta)=\rho_n^{-\alpha}f_n(z_n+\rho_n\zeta)$ locally uniformly converge to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$ with respect to the spherical metric in $\mathbb{P}^1$, which further satisfies an estimate