



ON GROWTH OF MEROMORPHIC SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS AND TWO CONJECTURES OF C.C. YANG*



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Abstract In this paper, we investigate the growth of the meromorphic solutions of the following nonlinear difference equations

$$f(z)^n + P_{n-1}(f) = 0,$$

where $n \geq 2$ and $P_{n-1}(f)$ is a difference polynomial of degree at most $n - 1$ in f with small functions as coefficients. Moreover, we give two examples to show that one conjecture proposed by Yang and Laine [2] does not hold in general if the hyper-order of $f(z)$ is no less than 1.

Key words growth; meromorphic solutions; difference equations; conjectures

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1 Introduction

In this paper, a meromorphic function always means meromorphic in the whole complex plane. We assume that the reader is familiar with the fundamental results and the standard notions of Nevanlinna's value distribution theory of meromorphic functions (see, e.g. [1, 4]). Let $f(z)$ be a meromorphic function. We use $\sigma(f)$ and $\lambda(f)$ to denote the order of growth and the exponent of convergence of zeros of a meromorphic function $f(z)$, respectively. In addition, we denote by $S(r, f)$ any quantity that satisfies the condition $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure. A meromorphic function $a(z) (\neq \infty)$ is called a small function with respect to $f(z)$ provided that $T(r, a(z)) = S(r, f)$. Moreover, the hyper-order of growth of $f(z)$ is defined as follows

$$\sigma_2(f) = \varlimsup_{r \rightarrow \infty} \frac{\log \log T(r, f)}{\log r}.$$

Recently, there was of Nevanlinna theory (see, e.g., [5, 6, 8–10, 12–17]). Given a meromorphic function $f(z)$ and a constant c , $f(z + c)$ is called a shift of f . As for a difference

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product, we mean a difference monomial of type $\prod_{j=1}^k f(z+c_j)^{n_j}$, where $c_1, \dots, c_k$ are complex constants, and $n_1, \dots, n_k$ are natural numbers. In the following, a difference polynomial, resp. a differential-difference polynomial, in f is defined as a finite sum of difference products of f and its shifts, resp. of products of f , derivatives of f and of their shifts, with all the coefficients of these monomials being small functions of f . Yang and Laine [2] investigated the nonlinear differential-difference equations and gave two conjectures on the nonexistence of entire solutions of infinite order to some differential-difference equations. We now recall Theorem 2.4 in [2] and the two conjectures.

Theorem 1.1 (see [2]) Let p, q be polynomials. Then a nonlinear difference equation

$$f(z)^2 + q(z)f(z+1) = p(z) \quad (1.1)$$

has no transcendental entire solutions of finite order.

Conjecture 1.2 (see [2]) There exists no entire function of infinite order that satisfies a difference equation of type

$$f(z)^n + q(z)f(z+1) = c \sin bz, \quad (1.2)$$

where q is a nonconstant polynomial, b, c are nonzero constants and $n \geq 2$ is an integer.

Conjecture 1.3 (see [2]) Let f be an entire function of infinite order and $n \geq 2$ be an integer. Then a differential-difference polynomial of the form $f^n + P_{n-1}(z, f)$ cannot be a nonconstant entire function of finite order, here $P_{n-1}(z, f)$ is a differential-difference polynomial of total degree at most $n-1$ in f , its derivatives and its shifts, with entire functions of finite order as coefficients. Moreover, we assume that all terms of $P_{n-1}(z, f)$ have total degree ≥ 1 .

Remark 1.4 Li [3] proved that Conjecture 1.3 is correct when the hyper-order of $f(z)$ is less than 1 by using a difference analogue of the lemma on the logarithmic derivative (see [5, 7]) which was extended to the case of hyper-order $\sigma_2(f) < 1$.

In this paper, we first investigate the following general nonlinear difference equation

$$f(z)^n + P_{n-1}(f) = 0, \quad (1.3)$$

where $n \geq 2$ and $P_{n-1}(f)$ is a difference polynomial of degree at most $n-1$ in f . We mainly focus on the growth of the transcendental meromorphic solutions of (1.3) and give the following Theorem 1.5 and Theorem 1.6.

Theorem 1.5 Suppose that $f(z)$ is a transcendental meromorphic solution of (1.3), and $P_{n-1}(f)$ is a difference polynomial of degree at most $n-1$ in f . Then $f(z)$ has infinite order.

Since all solutions of (1.3) is of infinite order, we now generalize Theorem 1.1 and estimate the hyper-order of the meromorphic solutions of (1.3) with rational coefficients.

Theorem 1.6 Suppose that $f(z)$ is a transcendental meromorphic solution of (1.3), and that all the coefficients of $P_{n-1}(f)$ are rational, and that all the shifts of $f(z)$ are $f(z+c_1), \dots, f(z+c_k)$. Denote $C = \max\{|c_1|, \dots, |c_k|\}$ and $m = \frac{n}{n-1}$.

(1) If $f(z)$ is entire or has finitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that

$$\log M(r, f) \geq Km^{r/C}$$

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