Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2016,**36B**(1):220–232
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

BLOW-UP OF CLASSICAL SOLUTIONS TO THE COMPRESSIBLE MAGNETOHYDRODYNAMIC EQUATIONS WITH VACUUM*

Shengguo ZHU (朱圣国)
Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China;
School of Mathematics, Georgia Tech Atlanta 30332, USA
E-mail: *zhushengguo@sjtu.edu.cn*

Abstract In this paper, we consider the formation of singularity for the classical solutions to compressible MHD equations without thermal conductivity or infinity electric conductivity when the initial data contains vacuum. We show that the life span of any smooth solution will not be extended to ∞, if the initial vacuum only appears in some local domain and the magnetic field vanishes on the interface that separates the vacuum and non-vacuum state, regardless the size of the initial data or the far field state.

Key words MHD; infinity electric conductivity; classical solutions; vacuum; blow-up

2010 MR Subject Classification 35A09; 35B44; 35M31; 35Q31; 35Q35; 35Q85

1 Introduction

Magnetohydrodynamics is that part of the mechanics of continuous media which studies the motion of electrically conducting media in the presence of a magnetic field. The dynamic motion of fluid and magnetic field interact strongly on each other, so the hydrodynamic and electrodynamic effects are coupled. The applications of magnetohydrodynamics cover a very wide range of physical objects, from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. In 3-D space, the compressible magnetohydrodynamic equations (MHD) in a domain $\Omega \subset \mathbb{R}^3$ can be written as

$$\begin{cases} H_t - \mathrm{rot}(u \times H) = -\mathrm{rot}\Big(\dfrac{1}{\sigma}\mathrm{rot}H\Big), \\ \mathrm{div}H = 0, \\ \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div}\mathbb{S} + \mathrm{rot}H \times H, \\ (\rho e)_t + \mathrm{div}(\rho e u) - \kappa\triangle\theta + P\mathrm{div}u = \mathrm{div}(u\mathbb{S}) - u\mathrm{div}\mathbb{S} + \dfrac{1}{\sigma}|\mathrm{rot}H|^2. \end{cases} \tag{1.1}$$

*Received September 29, 2014; revised January 11, 2015. The research was supported in part by National Natural Science Foundation of China (11231006), Natural Science Foundation of Shanghai (14ZR1423100) and China Scholarship Council.

The initial data is given by

$$(H,\rho,u,e)|_{t=0} = (H_0(x),\rho_0(x),u_0(x),e_0(x)),\ x\in\Omega. \tag{1.2}$$

In this system, $\Omega\subset\mathbb{R}^3$ is a smooth domain; $x\in\Omega$ is the spatial coordinate; $t\geq 0$ is the time; $H=(H_1,H_2,H_3)^\top$ is the magnetic field; $0<\sigma\leq\infty$ is the electric conductivity coefficient; ρ is the mass density; $u=(u_1,u_2,u_3)^\top\in\mathbb{R}^3$ is the velocity of fluids; e is the specific internal energy; $\kappa\geq 0$ is the thermal conductivity coefficient; $\mathbb{S}$ is the viscosity stress tensor given by

$$\mathbb{S}=\mu(\nabla u+(\nabla u)^\top)+\lambda \mathrm{div} u\mathbb{I}_3, \tag{1.3}$$

where $\mathbb{I}_3$ is the 3×3 unit matrix, μ is the shear viscosity coefficient, $\lambda+\frac{2}{3}\mu$ is the bulk viscosity coefficient, μ and λ are both real constants.

We only study the ideal polytropic fluids, so that P, e and θ are given by

$$P=R\rho\theta,\quad e=c_v\theta,\quad \gamma>1, \tag{1.4}$$

where θ is the absolute temperature, R and c_v are both positive constants, γ is the adiabatic index satisfying

$$R/c_v=\gamma-1.$$

Although the electric field E doesn't appear in system (1.1), it is indeed induced according to a relation

$$E=\frac{1}{\sigma}\mathrm{rot}H-u\times H$$

by moving the conductive flow in the magnetic field.

In the case that the domain Ω has boundary, the standard no-slip boundary condition or Navier-slip boundary condition will be supplemented.

In this paper, it will be always assumed that

$$\mu>0,\quad \lambda+\frac{2}{3}\mu>0,\quad \kappa=0,\quad 0<\sigma\leq+\infty. \tag{1.5}$$

When $H\equiv 0$ in 3-D space, the existence of unique local strong (or classical) solution with vacuum has been solved by many papers, and we refer the readers to [1, 2]. Huang-Li-Xin obtained the well-posedness of classical solutions with small energy but possibly large oscillations and vacuum for Cauchy problem in [7]. Some similar existence results was obtained for compressible MHD equations in [5, 8].

The finite time blow-up for the classical solutions with compactly supported initial density to compressible non-isentropic Navier-Stokes equations without thermal conductivity was proved in Xin [15], which was generalized by Cho-Kim [3] to the case of $\kappa>0$. Luo-Xin [12] proved the finite time blow-up of symmetric smooth solutions to two dimensional isentropic Navier-Stokes equations and analyzed the blow-up behavior at infinity time for one point vacuum initial data. Du-Li-Zhang [4] showed the finite time blow-up of smooth solutions to the isothermal case for the one dimensional case and two dimensional case with spherically symmetric assumptions.

If we remove the key assumption that the initial mass density is compactly supported, the finite time blow-up was proved in Rozanova [14] for classical solutions to compressible no-isentropic flow with highly decreasing at infinity. For the compressible isentroic Navier-Stokes or MHD equation in 3-D space, it can be shown in [13] as follows: