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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR GRADIENT SYSTEMS IN FINITE DIMENSIONAL SPACES*

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Abstract This paper deals with an abstract periodic gradient system in which the gradient is taken with respect to a variable metric. We obtain an existence and uniqueness result via the application of a global inverse theorem.

Key words existence and uniqueness; periodic solutions; gradient systems; global inverse theorem

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1 Introduction

In this paper, we investigate the existence and the uniqueness of solutions for the first order nonlinear periodic differential equation

$$\begin{cases} u \in W^{1,p}(0,T;\mathbb{R}^N), \\ u'(t) + \nabla_{g(t)} E(u(t)) = f(t) \text{ for a.e. } t \in (0,T), \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where $E : \mathbb{R}^N \to \mathbb{R}$ is a twice differentiable functional, $\nabla_{g(t)} E$ denotes the gradient of E with respect to a time-variable inner product $\langle \cdot, \cdot \rangle_{g(t)}$, $p \geq 2$ and $f \in L^p(0,T;\mathbb{R}^N)$.

The periodic boundary value problems were studied intensively in recent years under several assumptions on the functional E. Many methods and tools were used in order to solve these problems, e.g., fixed point methods, degree theory, variational methods, the upper-lower solutions method and perturbation and iterative techniques. We refer the reader to [7, 8, 11, 14, 16, 18–23, 32, 34] and the references therein for more details about these methods and for abstract results and their applications.

In this paper, the techniques of the proofs are based on the application of the following global inverse functions theorem.

Theorem 1.1 (see [24]) Let X and Y be two Banach spaces, and let $\phi : X \to Y$ be a map. The following assertions are equivalent:

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i) ϕ is a homeomorphism from X into Y,

ii) ϕ is a local homeomorphism and proper.

Recall that the map ϕ is proper if $\phi^{-1}(C)$ is a compact set in X whenever C is a compact set in Y.

Several authors obtained existence and uniqueness results for boundary value problems using global inversion theorems; see for example [1–3, 5, 6, 9, 12, 13, 15, 17, 26–30, 33, 35, 36].

In order to apply Theorem 1.1, we reformulate problem (1.1) as an algebraic equation

$$\phi(u) = v,$$

where $\phi : X \to Y$ is a mapping from a Banach space X into a Banach space Y, and we prove that ϕ satisfies assumptions of Theorem 1.1. Under some monotonicity condition on the derivative operator E' and under some nondegeneracy condition on g we prove that problem (1.1) admits a unique solution. In the setting of gradient systems in which the gradient is taken with respect to variable metrics, we refer to [10, Theorem 2.10 and Proof of Theorem 6.1-Part 1] in which the metric depends on the space variable, and to [4, Proof of Theorem 4-Part 1] in which the author considered metrics depending on the time and space variables. The problems considered in [4] and [10] are with initial data.

2 Functional Setting and Assumptions

First, we recall from [10] some basic facts and results about Euclidian and Riemannian metrics. Let $N \in \mathbb{N}^*$ and let $E : \mathbb{R}^N \to \mathbb{R}$ be a Fréchet differentiable functional. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the Euclidian inner product and norm on $\mathbb{R}^N$, respectively. The Euclidian gradient of E is the function ∇E which assigns to every point $u \in \mathbb{R}^N$ the unique element $\nabla E(u) \in \mathbb{R}^N$ such that

$$E'(u)v = \langle \nabla E(u), v \rangle, \ \forall\, v \in \mathbb{R}^N.$$

By the Riesz-Fréchet theorem, the euclidian gradient ∇E is well defined in the sense that it exists and it is unique. We denote by $\mathrm{Inner}(\mathbb{R}^{\mathrm{N}})$ the set of all inner products on $\mathbb{R}^N$. Let $T > 0$ and let $g : [0, T] \to \mathrm{Inner}(\mathbb{R}^{\mathrm{N}})$ be a function and denote by $\langle \cdot, \cdot \rangle_{g(t)}$ the inner product $g(t)$ at a time $t \in [0, T]$ and by $\| \cdot \|_{g(t)}$ the norm associated with this inner product. For every $t \in [0, T]$, the gradient of E with respect to the inner product $\langle \cdot, \cdot \rangle_{g(t)}$ is the function $\nabla_{g(t)} E$ which assigns to every point $u \in \mathbb{R}^N$ the unique element $\nabla_{g(t)} E(u) \in \mathbb{R}^N$ such that

$$E'(u)v = \langle \nabla_{g(t)} E(u), v \rangle_{g(t)}, \ \forall\, v \in \mathbb{R}^N.$$

By the Riesz-Fréchet theorem, the euclidian gradient $\nabla_{g(t)} E$ exists and is unique for every $t \in [0, T]$.

For every $t \in [0, T]$, let $Q(t) \in \mathcal{L}(\mathbb{R}^N)$ be defined by

$$\langle Q(t)v, w \rangle = \langle v, w \rangle_{g(t)}, \ \forall\, v, w \in \mathbb{R}^N.$$

Then we have

$$\nabla_{g(t)} E(u) = Q(t)^{-1} \nabla E(u), \ \forall\, t \in [0, T], \ \forall\, u \in \mathbb{R}^N.$$