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ON POINTS CONTAIN ARITHMETIC PROGRESSIONS IN THEIR LÜROTH EXPANSION*

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Abstract For any $x \in (0,1]$ (except at most countably many points), there exists a unique sequence $\{d_n(x)\}_{n\geq 1}$ of integers, called the digit sequence of x, such that

$$x = \sum_{j=1}^{\infty} \frac{1}{d_1(x)(d_1(x)-1)\cdots d_{j-1}(x)(d_{j-1}(x)-1)d_j(x)}.$$

The dexter infinite series expansion is called the Lüroth expansion of x. This paper is concerned with the size of the set of points x whose digit sequence in its Lüroth expansion is strictly increasing and contains arbitrarily long arithmetic progressions with arbitrary common difference. More precisely, we determine the Hausdorff dimension of the above set.

Key words Lüroth expansion; arithmetic progression; Hausdorff dimension

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1 Introduction

The Lüroth expansion was first introduced by Lüroth [1] in 1883. For any $x \in (0,1]$, the Lüroth map $T : (0,1] \to (0,1]$ is defined by

$$T(x) := d_1(x)(d_1(x)-1)\left(x - \frac{1}{d_1(x)}\right), \quad \text{where} \quad d_1(x) = \left[\frac{1}{x}\right] + 1. \tag{1.1}$$

Then we define the integer sequence $\{d_n(x)\}_{n\geq 1}$ by

$$d_n(x) = d_1\left(T^{n-1}(x)\right), \quad n \geq 1, \tag{1.2}$$

where T^n denotes the nth iterate of $T(T^0 = Id_{(0,1]})$.

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By algorithm (1.1) and (1.2), any $x \in (0,1]$(except at most countably many points) can be developed uniquely into an infinite series expansion of the form

$$x = \frac{1}{d_1(x)} + \sum_{j=2}^{n} \frac{1}{\prod\limits_{i=1}^{j-1} d_i(x)(d_i(x)-1)d_j(x)} + \frac{T^n(x)}{\prod\limits_{j=1}^{n} d_j(x)(d_j(x)-1)} \tag{1.3}$$

$$= \sum_{j=1}^{\infty} \frac{1}{d_1(x)(d_1(x)-1)\cdots d_{j-1}(x)(d_{j-1}(x)-1)d_j(x)}, \tag{1.4}$$

which is called the Lüroth expansion of x and denote it by $x = [d_1(x), d_2(x), \cdots, d_n(x), \cdots]$ for short.

The above algorithm implies $d_n \geq 2$ for all $n \geq 1$. On the contrary, for a sequence of integers $\{d_n\}_{n\geq 1}$ satisfying $d_n \geq 2$, $\forall\ n \geq 1$, there exists a unique $x \in (0,1]$ such that $d_n(x) = d_n$, $\forall\ n \in \mathbb{N}$ in the Lüroth expansion of x. Namely, each irrational $x \in (0,1]$ is corresponding to an infinite integer sequence $\{d_n\}_{n\geq 1}$.

Whether integer subset contains arbitrarily long arithmetic progressions is a long-standing question in number theory, especially for some peculiar subsets, such as primes. For this, Van der Waerden [2] in 1927 established that while the set of integers is arbitrarily partitioned into two classes, at least one class contains arbitrarily long arithmetic progressions. In 2008, Green and Tao [3] gave a affirmative answer to the problem that the primes contain arbitrarily long arithmetic progressions.

Since $\{d_n\}_{n\geq 1}$ can assume arbitrarily large values, it is possible that there are points whose sequence of digits in its Lüroth expansion contains arbitrarily long arithmetic progressions. In this paper, we discuss the above long-standing question in the setting of Lüroth expansion in the view of metric number theory. To be specific, we are interested in the set of points whose sequence of digits in its Lüroth expansion is strictly increasing and contains arbitrarily long arithmetic progressions. Denote such a set by E_S, i.e.,

$$E_S = \{x \in (0,1] \colon \{d_n(x)\}_{n\geq 1} \text{ is strictly increasing and contains arbitrarily long arithmetic progressions}\}.$$

Furthermore, we care about the points whose sequence of digits in its Lüroth expansion contains the arithmetic progressions with arbitrary common difference as well as satisfying above properties. More precisely, we consider the set E_{AS} defined as follows:

$$E_{AS} = \{x \in (0,1] \colon \{d_n(x)\}_{n\geq 1} \text{ is strictly increasing and contains arbitrarily long arithmetic progressions with arbitrary common difference}\}.$$

It is natural to ask how large such sets are in the sense of Lebesgue measure or Hausdorff dimension. We prove that

Theorem 1.1 $\dim_H E_S = \dim_H E_{AS} = \frac{1}{2}$, where $\dim_H$ denotes the Hausdorff dimension.

The growth speed of the digit sequence $\{d_n(x)\}_{n\geq 1}$ was studied in [4]. The metric and ergodic properties of the digit sequence $\{d_n(x)\}_{n\geq 1}$ and the Lüroth map T defined by (1.1) were extensively studied in [5] (see also [6–11]). The behavior of approximating real numbers by Lüroth expansion was thoroughly investigated in [12–14]. Since the Lüroth system can also be viewed as an infinite iterated function system, dimensional theory in Lüroth expansion