Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2016,**36B**(2):317–324
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

MEAN-FIELD LIMIT OF BOSE-EINSTEIN CONDENSATES WITH ATTRACTIVE INTERACTIONS IN $\mathbb{R}^{2*}$

Yujin GUO (郭玉劲)
Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China
E-mail: *yjguo@wipm.ac.cn*

Lu LU (陆璐)†
School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430073, China
E-mail: *lulu@znufe.edu.cn*

Abstract Starting with the many-body Schrödinger Hamiltonian in $\mathbb{R}^2$, we prove that the ground state energy of a two-dimensional interacting Bose gas with the pairwise attractive interaction approaches to the minimum of the Gross-Pitaevskii energy functional in the mean-field regime, as the particle number $N \to \infty$ and however the scattering length $\kappa \to 0$. By fixing $N|\kappa|$, this leads to the mean-field approximation of Bose-Einstein condensates with attractive interactions in $\mathbb{R}^2$.

Key words Bose-Einstein condensation; attractive interactions; Gross-Pitaevskii functional; mean-field approximation

2010 MR Subject Classification 35Q40; 46N50; 82D50

1 Introduction

As the experimental realization of Bose-Einstein condensates (BEC) in 1995 (cf. [1, 9]), BEC has been investigated intensively over the past few years. The forces between the atoms in BEC can be either attractive or repulsive. In contrast to the repulsive case, the system of the attractive case collapses if the particle number increases beyond a critical value, seeing, for example, [15, 17, 18, 30] or [8, Sec. III.B], which gives the existence of a critical particle number for cold atoms. The repulsive case has been analyzed widely over the past few years; see, for example, [21, 23–25] and references therein. In view of this fact, we shall focus on the attractive case in this article.

*Received August 6, 2014; revised October 12, 2015. This work is partially supported by the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry of China, and National Center for Mathematics and Interdisciplinary Sciences in China.

†Corresponding author.

As illustrated in [4, 8, 11, 30] and references therein, BEC with attractive interactions in two dimensions can be described by the following constraint minimization problem

$$E_a(N) = \inf\Big\{\mathscr{E}(\rho) \ \Big|\ \rho \geq 0,\ \ \rho^{\frac{1}{2}} \in \mathcal{H} \text{ and } \int_{\mathbb{R}^2} \rho \mathrm{d}x = N\Big\}, \tag{1.1}$$

where $N > 0$ denotes the particle number of cold atoms, and the Gross-Pitaevskii (GP) energy functional $\mathscr{E}(\rho)$ is of the form

$$\mathscr{E}(\rho) = \Big(\rho^{\frac{1}{2}}, \Big\{-\Delta + V(x)\Big\}\rho^{\frac{1}{2}}\Big) - \frac{a}{2N}\int_{\mathbb{R}^2} \rho^2 \mathrm{d}x, \quad \rho^{\frac{1}{2}} \in \mathcal{H}. \tag{1.2}$$

Here, $\frac{a}{N} > 0$ denotes the scattering length[1] κ of the (attractive) interaction potential, that is,

$$\kappa := \frac{a}{N} > 0, \tag{1.3}$$

and from the physical point of view, the trapping potential $V(x)$ is assumed to satisfy

(V). $V(x) \geq 0,\ V(x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ and $\lim\limits_{|x|\to\infty} V(x) = \infty$,

so that $\mathcal{H}$ is defined as

$$\mathcal{H} := \Big\{\rho^{\frac{1}{2}} \in H^1(\mathbb{R}^2) \ \Big|\ \int_{\mathbb{R}^2} V(x)\rho(x)\mathrm{d}x < \infty\Big\}. \tag{1.4}$$

Alternatively, it is convenient to consider the L^2-normalized minimization problem

$$e(a) = \inf\Big\{\mathcal{E}(\rho) \ \Big|\ \rho \geq 0,\ \rho^{\frac{1}{2}} \in \mathcal{H} \text{ and } \int_{\mathbb{R}^2} \rho \mathrm{d}x = 1\Big\}, \tag{1.5}$$

where the GP energy functional $\mathcal{E}(\rho)$ satisfies

$$\mathcal{E}(\rho) = \Big(\rho^{\frac{1}{2}}, \Big\{-\Delta + V(x)\Big\}\rho^{\frac{1}{2}}\Big) - \frac{a}{2}\int_{\mathbb{R}^2} \rho^2 \mathrm{d}x, \quad \rho^{\frac{1}{2}} \in \mathcal{H}, \tag{1.6}$$

and the physical constant $a > 0$ is the same as that of (1.2). One can then check that for $\bar\rho(x) = N\rho(x)$,

$$\mathscr{E}(\bar\rho) = N\mathcal{E}(\rho) \text{ and } E_a(N) = Ne(a), \tag{1.7}$$

which implies that the analysis of $E_a(N)$ and $e(a)$ can be reduced to each other.

The analytic properties of $e(a)$ (and equivalently of $E_a(N)$) were studied recently in [15, 16]. It actually turns out that the problem $e(a)$ is related closely to the following nonlinear scalar field equation

$$-\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^2, \text{ where } u \in H^1(\mathbb{R}^2). \tag{1.8}$$

Remark from [13, 19, 20] that, up to translations, (1.8) admits a unique positive radially symmetric solution, which we denote $Q = Q(|x|)$. Note also from [13, Prop. 4.1] that $Q(|x|)$ has the following exponential decay,

$$Q(|x|),\ |\nabla Q(|x|)| = O(|x|^{-\frac{1}{2}}e^{-|x|}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

Moreover, we recall from [29] the following Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^2} |u(x)|^4 \mathrm{d}x \leq \frac{2}{\|Q\|_2^2}\int_{\mathbb{R}^2} |\nabla u(x)|^2 \mathrm{d}x \int_{\mathbb{R}^2} |u(x)|^2 \mathrm{d}x, \quad u \in H^1(\mathbb{R}^2), \tag{1.10}$$

[1]From the physical point of view, the scattering length κ of attractive BEC is negative. Here, we use $\kappa > 0$ for convenience.

Download English Version:

https://daneshyari.com/en/article/4663496

Download Persian Version:

https://daneshyari.com/article/4663496

Daneshyari.com