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DIFFERENTIAL OPERATORS OF INFINITE ORDER IN THE SPACE OF RAPIDLY DECREASING SEQUENCES*

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Abstract We consider the space of rapidly decreasing sequences s and the derivative operator D defined on it. The object of this article is to study the equivalence of a differential operator of infinite order; that is $\varphi(D) = \sum\limits_{k=0}^{\infty} \varphi_k D^k$. φ_k constant numbers an a power of D. D^n, meaning, is there a isomorphism X (from s onto s) such that $X\varphi(D) = D^n X$?. We prove that if $\varphi(D)$ is equivalent to D^n, then $\varphi(D)$ is of finite order, in fact a polynomial of degree n. The question of the equivalence of two differential operators of finite order in the space s is addressed too and solved completely when $n = 1$.

Key words Ordinary differential operators; sequence spaces; operators on function spaces

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1 Introduction

The problem of the equivalence of differential operators was treated by many authors in different contexts. The works of Delsarte and Lions [1–5] are well-known where they introduced the notion of "operateur de transmutation". Later on, in the 1960s, the term "equivalence of operators" appeared, being the subject intensively studied [6–23].

Let us recall the concept of "operateur de transmutation". If T and S are two diferential operators in a space H, an operator X is an "operateur de transmutation" from T to S when X is an isomorphism from H onto H such that $SX = XT$.

The operators T and S are said to be equivalent when there is an "operateur de transmutation" from T to S; that is, there an isomorphism X such that $SX = XT$.

If T and S are differential operators and H a space of real variable functions, the problem is solved only in particular cases and it seems that it is difficult to give a general solution as Delsarte and Lions pointed out as far back as 1957. On the contrary, the situation when T and S are differential operators in the complex plane and H is a space of functions of a complex variable is very different; in fact, when H is the space of entire functions of a complex variable and T and S are differential operators of the same order with constant coefficients, then T and S

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are equivalent. Several authors treated the question in spaces of analytic functions considering differential operators with or without singularities as well as differential operators of infinite order.

Nagnibida N. I. and Oliinyk N. P [21] studied the equivalence of a differential operator of infinite order to a power D^n of the usual derivative in analytic spaces. They proved that the only differential operators of infinite order equivalent to D^n are differential operators of finite order; that is $p_n(D) = \sum\limits_{k=0}^{n} \varphi_k D^k$, $\varphi_k \in \mathbb{C}$, $|\varphi_n| = 1$ (in the space of analytic functions on a finite disc) and $p_n(D) = \sum\limits_{k=0}^{n} \varphi_k D^k$, $\varphi_n \in \mathbb{C}$, $|\varphi_n| \neq 0$ (in the space of analytic functions on the whole complex plane).

As the spaces of analytic complex functions can be considered as sequence spaces, it seems natural to study the equivalence of differential operators on sequence spaces. As the results depend greatly, not only on the differentials operators, but on the sequence spaces taken, we think it convenient to work with a concrete sequence space. A sequence space that plays an important role in mathematics is the space of rapidly decreasing sequences s. It is enough to observe that very well known space functions as $C^{\infty}_{2\pi}(\mathbb{R}) = \{2\pi\text{-periodic, } \mathcal{C}^{\infty}\text{-funcions on } \mathbb{R}\}$, $S(\mathbb{R})$ (the Schwartz space), $\mathcal{D}[a,b] = \{f \in C^{\infty}(\mathbb{R}), \operatorname{Supp}(f) \subset [a,b]\}$, $\mathcal{C}^{\infty}[a,b] = \{f : [a,b] \to \mathbb{C}, f \text{ infinitely differentiable}\}$ are isomorphic to s [24]. Using this fact, it is easy to see that two differential operators of order one with constant coefficients are not equivalent on the space $C^{\infty}_{2\pi}(R)$.

In this article, we consider a continuous differentiable operator of infinite order $\varphi(D)$ with constant coefficients in the sequence space s and prove that, as in the case of analytic functions, its equivalence to a power of the usual derivative D^n implies that it is of finite order, in fact, a polynomial of degree n, $p_n(D) = \sum\limits_{k=0}^{n} \varphi_k D^k$. We consider, too, the problem of the equivalence of two differential operators of finite order in the space s not having, for $n > 1$, a close result. For $n = 1$, a differential operator of order one is equivalent to D if and only if $|\varphi_1| = 1$.

2 Terminology

An infinite power series space $\Lambda_\infty(\alpha)$ (respectively a finite power series space $\Lambda_1(\alpha)$) is the sequence space

$$\Lambda_\infty(\alpha) = \left\{(x_n) : \sum_{n=0}^{\infty} |x_n|\, e^{k\alpha_n} < \infty, \quad k \in N\right\},$$
$$\Lambda_1(\alpha) = \left\{(x_n) : \sum_{n=0}^{\infty} |x_n|\, e^{-\frac{1}{k}\alpha_n} < \infty, \quad k \in N\right\},$$

where (α_n) is an increasing sequence of positive real numbers going to infinity. They are Fréchet spaces with a canonical basis (e_n), the topology given by the norms

$$\|(x_n)\|_k = \sum_{n=0}^{\infty} |x_n| e^{k\alpha_n},$$
$$\|(x_n)\|_k = \sum_{n=0}^{\infty} |x_n| e^{-\frac{1}{k}\alpha_n}.$$