Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2016,**36B**(2):443–452
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

RECURRENCE FOR WEIGHTED TRANSLATIONS ON GROUPS*

Chung-Chuan CHEN (陈中川)
Department of Mathematics Education, National Taichung University of Education, Taiwan
E-mail: *chungchuan@mail.ntcu.edu.tw*

Abstract Let G be a locally compact group, and let $1 \leq p < \infty$. We characterize topologically multiply recurrent weighted translation operators on $L^p(G)$ in terms of the Haar measure and the weight function. We also show that there do not exist any recurrent weighted translation operators on $L^\infty(G)$.

Key words Topologically multiple recurrence; recurrence; hypercyclicity; locally compact group; L^p-space

2010 MR Subject Classification 37B20; 47A16; 43A15

1 Introduction

Recently, we gave sufficient and necessary conditions for weighted translation operators on groups to be hypercyclic and chaotic in [1–4], which subsumes some works in [5–8]. The notion of hypercyclicity in linear dynamics is close to, but stronger than the notion of recurrence in topological dynamics in [9]. It is well known that every hypercyclic operator is recurrent on separable Banach spaces in [10]. However, this is not the case for topologically multiple recurrence, which is a stronger notion than recurrence. There exists a hypercyclic weighted backward shift on $\ell^2(\mathbb{Z})$ in [10], which is not topologically multiply recurrent. In this note, we will give a sufficient and necessary condition for weighted translation operators on the Lebesgue space L^p $(1 \leq p < \infty)$ of a locally compact group to be topologically multiply recurrent in terms of the Haar measure and the weight function, and show, on the L^∞ space, there are no recurrent weighted translation operators.

In linear dynamics, we first recall that an operator T on a Banach space X is called hypercyclic if there exists a vector $x \in X$ such that its orbit under T denoted by $\mathrm{Orb}(x,T) := \{x, Tx, T^2x, \cdots\}$ is dense in X in which x is said to be a hypercyclic vector of T. It is known that hypercyclicity is equivalent to topological transitivity. An operator T is topologically transitive if given two nonempty open subsets $U, V \subset X$, there is some $n \in \mathbb{N}$ such that $T^nU \cap V \neq \emptyset$. If $T^nU \cap V \neq \emptyset$ from some n onwards, then T is called topologically mixing. Hypercyclicity and transitivity have been studied by many authors. We refer to these books [11, 12] on this subject. In topological dynamics, an operator T is topologically multiply

*Received May 9, 2014; revised June 17, 2015. The author is supported by MOST of Taiwan (MOST 104-2115-M-142-002-).

recurrent if for every positive integer N and every nonempty open set U in X, there is some $n \in \mathbb{N}$ such that $U \cap T^{-n}U \cap T^{-2n}U \cap \cdots \cap T^{-Nn}U \neq \emptyset$. If $N = 1$, then T is called recurrent, that is, the condition $U \cap T^{-n}U \neq \emptyset$ is satisfied.

The motivation to connect hypercyclicity with recurrence is inspired by the works in [9, 10]. In [10], Costakis and Parissis characterized topologically multiply recurrent weighted shifts on $\ell^p(\mathbb{Z})$ in terms of the weight sequence. On the other hand, that the space $\ell^\infty(\mathbb{Z})$ does not support any recurrent weighted shifts was shown in [9]. We note that the weighted shifts on $\ell^p(\mathbb{Z})$ and $\ell^\infty(\mathbb{Z})$ are a special case of the weighted translation operators on the Lebesgue space of a locally compact group. In this article, we will extend some results in [9, 10] to the setting of translations on groups.

In what follows, let G be a locally compact group with identity e and a right-invariant Haar measure λ. We denote by $L^p(G)$ $(1 \le p \le \infty)$ the complex Lebesgue space, with respect to λ. A bounded function $w : G \to (0, \infty)$ is called a weight on G. Let $a \in G$ and let δ_a be the unit point mass at a. A weighted translation on G is a weighted convolution operator $T_{a,w} : L^p(G) \longrightarrow L^p(G)$ defined by

$$T_{a,w}(f) = wT_a(f) \qquad (f \in L^p(G)),$$

where w is a weight on G and $T_a(f) = f * \delta_a \in L^p(G)$ is the convolution:

$$(f * \delta_a)(x) := \int_G f(xy^{-1})\mathrm{d}\delta_a(y) = f(xa^{-1}) \qquad (x \in G).$$

If $w^{-1} \in L^\infty(G)$, then the weighted translation operator $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$ is the inverse of $T_{a,w}$. We write $S_{a,w}$ for $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$ to simplify notation. We assume $w, w^{-1} \in L^\infty(G)$ throughout.

2 Recurrence on $L^p(G)$

In this section, we will prove the result on $L^p(G)(1 \le p < \infty)$ for translations by aperiodic elements in G. An element a in a group G is called a torsion element if it is of finite order. In a locally compact group G, an element $a \in G$ is called periodic (or compact) in [4] if the closed subgroup $G(a)$ generated by a is compact. We call an element in G aperiodic if it is not periodic. For discrete groups, periodic and torsion elements are identical. It is proved in [4] that an element $a \in G$ is aperiodic if and only if for any compact set $K \subset G$, there exists some $N \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > N$.

We will make use of the aperiodic condition to obtain the result below. We note that [4] in many familiar non-discrete groups, including the additive group $\mathbb{R}^d$, the Heisenberg group, and the affine group, all elements except the identity are aperiodic.

Theorem 2.1 Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. The following conditions are equivalent.

(i) $T_{a,w}$ is topologically multiply recurrent;

(ii) For each $N \in \mathbb{N}$ and each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences (for $1 \le l \le N$)

$$\varphi_{ln} := \prod_{s=1}^{ln} w * \delta_{a^{-1}}^s \quad \text{and} \quad \widetilde{\varphi}_{ln} := \left(\prod_{s=0}^{ln-1} w * \delta_a^s\right)^{-1}$$