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SOME PROPERTIES OF OPERATOR-VALUED FRAMES*

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Abstract Operator-valued frames (or g-frames) are generalizations of frames and fusion frames and have been used in packets encoding, quantum computing, theory of coherent states and more. In this article, we give a new formula for operator-valued frames for finite dimensional Hilbert spaces. As an application, we derive in a simple manner a recent result of A. Najati concerning the approximation of g-frames by Parseval ones. We obtain also some results concerning the best approximation of operator-valued frames by its alternate duals, with optimal estimates.

Key words Frames; g-frames

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1 Introduction

Frames in Hilbert spaces were introduced by Duffin and Schaeffer [10] in 1952, in the context of nonharmonic Fourier series. After a couple of years, in 1986, frames were brought to life by Daubechies, Grossman, and Meyer [9]. Frames have nice properties which makes them useful tools in signal processing, image processing, coding theory, sampling theory and more.

In the following, we denote by $\mathcal{H}$ a separable Hilbert space and by $\mathcal{L}(\mathcal{H})$ the space of all linear bounded operators on $\mathcal{H}$.

Definition 1.1 A family of elements $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \le \sum_{n=1}^{\infty} |\langle x, f_n\rangle|^2 \le B\|x\|^2, \quad x \in \mathcal{H}.$$

The constants A, B are called frame bounds.

We say that a frame is tight if $A = B$, a Parseval frame if $A = B = 1$, and an exact frame if it ceases to be a frame when any one of its elements is removed.

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The exact frames are in fact Riesz bases. If just the last inequality in the above definition holds, we say that $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence.

When $\{f_n\}_{n=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$, the operator

$$T : l^2 \to \mathcal{H},\ \ T\{c_n\}_{n=1}^{\infty} := \sum_{n=1}^{\infty} c_n f_n$$

is called synthesis operator(or pre-frame operator). Its adjoint operator is given by

$$\Theta = T^* : \mathcal{H} \to l^2,\ \ \Theta x = \{\langle x, f_n\rangle\}_{n=1}^{\infty},$$

and is called the analysis operator. By composing T with its adjoint T^*, we obtain the frame operator

$$S : \mathcal{H} \to \mathcal{H},\ \ Sx = TT^*x = \sum_{n=1}^{\infty} \langle x, f_n\rangle f_n.$$

The next theorem is one of the most important results about frames.

Theorem 1.2 Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}$ be a frame for $\mathcal{H}$ with frame operator S. Then

(i) S is invertible and self-adjoint;

(ii) every $x \in \mathcal{H}$ can be represented as

$$x = \sum_{n=1}^{\infty} \langle x, f_n\rangle S^{-1} f_n = \sum_{n=1}^{\infty} \langle x, S^{-1} f_n\rangle f_n. \tag{1.1}$$

The relation (1.1) is called the reconstruction formula. We call $\{\langle x, S^{-1} f_n\rangle\}_{n=1}^{\infty}$ the frame coefficients.

The frame $\{S^{-1} f_n\}_{n=1}^{\infty}$ is called the canonical dual frame of $\{f_n\}_{n=1}^{\infty}$. A sequence $\{g_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ is called an alternate dual for $\{f_n\}_{n=1}^{\infty}$ if it satisfies the following equality

$$x = \sum_{n=1}^{\infty} \langle x, g_n\rangle f_n, \quad \forall x \in \mathcal{H}.$$

A generalization of frames, which allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space, was obtain by L. Găvruţa [13].

In 2006, W. Sun [21] introduced the concept of g-frame. g-frames are generalized frames, which include ordinary frames, bounded invertible linear operators, fusion frames, as well as many recent generalizations of frames. See also the article of V. Kaftal, D. Larson and S. Zhang [18]. For the general theory of fusion frames, see the article of P.G. Casazza et al [6] and P. Găvruţa [16].

For the connection between the theory of g-frames and quantum theory as in [7, 19], see the papers [1] and [17].

In the following, we consider $\mathcal{H}$ and $\mathcal{K}$ to be two Hilbert spaces. We denote by $\mathcal{L}(\mathcal{H}, \mathcal{K})$ the space of all linear bounded operators from $\mathcal{H}$ into $\mathcal{K}$. By $\mathbb{I}$ we denote a finite or a countable set.

Definition 1.3 We say that a sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{K}) : i \in \mathbb{I}\}$ is a generalized frame or a g-frame for $\mathcal{H}$ if there exist two positive constants A and B such that

$$A\|x\|^2 \le \sum_{i\in\mathbb{I}} \|\Lambda_i x\|^2 \le B\|x\|^2, \quad \forall x \in \mathcal{H}.$$

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