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DISCRETE GALERKIN METHOD FOR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract In this article, we develop a fully Discrete Galerkin(DG) method for solving initial value fractional integro-differential equations(FIDEs). We consider Generalized Jacobi polynomials(GJPs) with indexes corresponding to the number of homogeneous initial conditions as natural basis functions for the approximate solution. The fractional derivatives are used in the Caputo sense. The numerical solvability of algebraic system obtained from implementation of proposed method for a special case of FIDEs is investigated. We also provide a suitable convergence analysis to approximate solutions under a more general regularity assumption on the exact solution. Numerical results are presented to demonstrate the effectiveness of the proposed method.

Key words Fractional integro-differential equation(FIDE); Discrete Galerkin(DG); Generalized Jacobi Polynomials(GJPs); Caputo derivative

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1 Introduction

In this article, we provide a convergent numerical scheme for solving FIDE

$$\begin{cases} \mathcal{D}^q u(x) = p(x)u(x) + f(x) + \lambda \displaystyle\int_0^x K(x,t)u(t)\mathrm{d}t, \quad x \in \Omega = [0,1], \\ u(0) = 0, \end{cases} \tag{1.1}$$

where $q \in \mathbb{R}^+ \bigcap (0,1)$. The symbol $\mathbb{R}^+$ is the collection of all positive real numbers. $p(x)$ and $f(x)$ are given continuous functions and $K(x,t)$ is a given sufficiently smooth kernel function, and $u(x)$ is the unknown function.

Noting that the condition $u(0) = 0$ is not restrictive, due to the fact that (1.1) with non-homogeneous initial condition $u(0) = d,\ d \neq 0$ can be converted to the following homogeneous FIDE

$$\begin{cases} \mathcal{D}^q \tilde{u}(x) = p(x)\tilde{u}(x) + \tilde{f}(x) + \lambda \displaystyle\int_0^x K(x,t)\tilde{u}(t)\mathrm{d}t, \quad x \in \Omega = [0,1], \\ \tilde{u}(0) = 0, \end{cases}$$

by the simple transformation $\tilde{u}(x) = u(x) - d$, where

$$\tilde{f}(x) = f(x) + d\left(p(x) + \lambda \int_0^x K(x,t)\mathrm{d}t\right).$$

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Such kind of equations arise in the mathematical modeling of various physical phenomena, such as heat conduction, materials with memory, combined conduction, convection and radiation problems ([3, 5, 29, 30]).

$\mathcal{D}^q u(x)$ denotes the fractional Caputo differential operator of order q and is defined as([8, 19, 31])

$$\mathcal{D}^q u(x) = \mathcal{I}^{1-q} u'(x), \tag{1.2}$$

where

$$\mathcal{I}^\mu u(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-s)^{\mu-1} u(s) \mathrm{d}s \tag{1.3}$$

is the fractional integral operator from order μ. $\Gamma(\mu)$ is the well known Gamma function. The following relation holds[8]

$$\mathcal{I}^q(\mathcal{D}^q u(x)) = u(x) - u(0). \tag{1.4}$$

From the relation above, it is easy to check that (1.1) is equivalent to the following weakly singular Volterra integral equation

$$u(x) = g(x) + \lambda \int_0^x \bar{K}(x,t) u(t) \mathrm{d}t. \tag{1.5}$$

Here, $g(x) = \mathcal{I}^q f(x)$ and $\bar{K}(x,t) = \frac{(x-t)^{q-1}}{\Gamma(q)} p(t) + \int_t^x \frac{(x-s)^{q-1}}{\Gamma(q)} K(s,t) \mathrm{d}s$. From the well known existence and uniqueness Theorems ([4, 35]), it can be concluded that if the following conditions are fulfilled:

- $f(x) \in C^l(\Omega), \quad l \geq 1,$
- $p(x) \in C^l(\Omega), \quad l \geq 1,$
- $K(x,t) \in C^l(D), \quad D = \{(x,t); 0 \leq t \leq x \leq 1\}, \quad l \geq 1,$
- $K(x,x) \neq 0,$

then the regularity of the unique solution $u(x)$ of (1.5) and also (1.1) is described by

$$u(x) = \sum_{(j,k)} \gamma_{j,k} x^{j+kq} + U_l(x;q) \in C^l(0,1] \bigcap C(\Omega), \quad \text{with} \quad |u'(x)| \leq C_q x^{q-1}, \tag{1.6}$$

where the coefficients $\gamma_{j,k}$ are some constants, $U_l(.;q) \in C^l(\Omega)$, and $(j,k) := \{(j,k) : \quad j,k \in \mathbb{N}_0, \ j+kq < l\}$. Here, $\mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, where the symbol $\mathbb{N}$ denotes the collection of all natural numbers. Thus, we must expect the first derivative of the solution to has a discontinuity at the origin. More precisely, if the given functions $g(x), p(x)$, and $K(x,t)$ are real analytic in their domains, then it can be concluded that there is a function $U = U(z_1, z_2)$ that is real and analytic at $(0,0)$, so that solutions of (1.5) and also (1.1) can be written as $u(x) = U(x, x^q)$ ([4, 35]).

Recently, several numerical methods for the numerical solution of FIDE's were proposed; see [2, 11, 12, 15, 20–28, 33, 36, 38, 39]. In [2], an analytical solution for a class of FIDE's was proposed. In [11], authors applied collocation method to solve the nonlinear FIDE's. In [15], Taylor expansion approach was presented for solving a class of linear FIDE's including those of Fredholm and Volterra types. In [20], Chebyshev pseudospectral method was implemented to solve linear and nonlinear system of FIDE's. Adomian decomposition method to solve nonlinear FIDE's was proposed in [21]. In [22], authors solved FIDE's by adopting hybrid collocation method to an equivalent integral equation of convolution type. In [23], authors proposed an analyzed spectral Jacobi collocation method for the numerical solution of general linear FIDE's.