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# FREDHOLM OPERATORS ON THE SPACE OF BOUNDED SEQUENCES *

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**Abstract** Necessary and sufficient conditions are studied that a bounded operator $Tx = (x_1^*x, x_2^*x, \cdots)$ on the space $\ell_\infty$, where $x_n^* \in \ell_\infty^*$, is lower or upper semi-Fredholm; in particular, topological properties of the set $\{x_1^*, x_2^*, \cdots\}$ are investigated. Various estimates of the defect $d(T) = \operatorname{codim} R(T)$, where $R(T)$ is the range of $T$, are given. The case of $x_n^* = d_n x_{t_n}^*$, where $d_n \in \mathbb{R}$ and $x_{t_n}^* \geq 0$ are extreme points of the unit ball $B_{\ell_\infty^*}$, that is, $t_n \in \beta\mathbb{N}$, is considered. In terms of the sequence $\{t_n\}$, the conditions of the closedness of the range $R(T)$ are given and the value $d(T)$ is calculated. For example, the condition $\{n : 0 < |d_n| < \delta\} = \emptyset$ for some $\delta$ is sufficient and if for large $n$ points $t_n$ are isolated elements of the sequence $\{t_n\}$, then it is also necessary for the closedness of $R(T)$ ($t_{n_0}$ is isolated if there is a neighborhood $\mathcal{U}$ of $t_{n_0}$ satisfying $t_n \notin \mathcal{U}$ for all $n \neq n_0$). If $\{n : |d_n| < \delta\} = \emptyset$, then $d(T)$ is equal to the defect $\delta\{t_n\}$ of $\{t_n\}$. It is shown that if $d(T) = \infty$ and $R(T)$ is closed, then there exists a sequence $\{A_n\}$ of pairwise disjoint subsets of $\mathbb{N}$ satisfying $\chi_{A_n} \notin R(T)$.

**Key words** Fredholm operator; space $\ell_\infty$; Stone-Čech compactification $\beta\mathbb{N}$

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## 1 Introduction

Let $X$ and $Y$ be Banach spaces. As usual, $\mathcal{L}(X,Y)$ will denote the space of all (bounded, linear) operators and $\mathcal{K}(X,Y)$ the space of all compact operators from $X$ into $Y$.

**Definition 1.1** For an operator $T \in \mathcal{L}(X,Y)$, the null space $N(T)$ and the range $R(T)$ of $T$ [1, p. 155] are the sets

$$N(T) = \{x \in X : Tx = 0\} \quad \text{and} \quad R(T) = \{Tx : x \in X\}.$$

The dimension of $N(T)$ is called the nullity of $T$ and is denoted $n(T)$. The codimension of $R(T)$ is called the defect of $T$ and is denoted $d(T)$.

**Definition 1.2** An operator $T$ is said to be Fredholm [1, p. 156] if its nullity $n(T)$ and defect $d(T)$ are both finite.

The collection of all Fredholm operators from $X$ into $Y$ will be denoted by $\Phi(X,Y)$; as usual, we shall write $\Phi(X)$ instead of $\Phi(X,X)$ (analogously, for example, for $\mathcal{L}(X)$ and $\mathcal{K}(X)$).

One of most important characterizations of Fredholm operators is the following Atkinson's theorem [1, p. 161] establishing the connection between Fredholm and compact operators: an

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operator $T$ from $X$ into $Y$ is Fredholm if and only if there exist operators $K_1 \in \mathcal{K}(X)$, $K_2 \in \mathcal{K}(Y)$, and $S_1, S_2 \in \mathcal{L}(Y, X)$ such that

$$S_1 T = I_X - K_1 \quad \text{and} \quad T S_2 = I_Y - K_2. \tag{1.1}$$

Now, let us consider the Banach space $\ell_\infty$ of all bounded sequences equipped with the sup-norm. Every operator $T$ on $\ell_\infty$ can be represented in the form

$$Tx = (x_1^* x, x_2^* x, \cdots), \tag{1.2}$$

where the sequence $\{x_n^*\}$ of functionals in the dual $\ell_\infty^*$ of the space $\ell_\infty$ is uniquely defined and bounded. Conversely, every bounded sequence $\{x_n^*\}$ in $\ell_\infty^*$ defines an operator $T$ on $\ell_\infty$ via the formula (1.2). In this case, we shall say that a sequence $\{x_n^*\}$ defines an operator $T$.

On the other hand, an operator $T$ on $\ell_\infty$ defined by a sequence $\{x_n^*\}$ is compact [3] if and only if the set $\{x_1^*, x_2^*, \cdots\}$ is relatively compact in $\ell_\infty^*$. Therefore, the compactness of the operator $T$ can be characterized in terms of geometrical properties of the set $\{x_1^*, x_2^*, \cdots\}$. Taking into account the connection which exists between Fredholm and compact operators (see (1.1)), the hypothesis arises that Fredholm operators on $\ell_\infty$ also can be characterized in terms of geometrical properties of the set $\{x_1^*, x_2^*, \cdots\}$. The main purpose of this article is to obtain some results in this direction.

**Definition 1.3** A bounded sequence $\{x_n^*\}$ of functionals in $\ell_\infty^*$ is said to be Fredholm if the operator $T$ defined by this sequence is Fredholm.

We also recall the following two notions (see [2, p. 33]).

**Definition 1.4** An operator $T \in \mathcal{L}(X, Y)$ is said to be upper semi-Fredholm if $n(T) < \infty$ and $R(T)$ is closed and is said to be lower semi-Fredholm if $d(T) < \infty$.

We mention at once that by Kato's theorem [1, p. 156], the relation $d(T) < \infty$ implies the closedness of $R(T)$. The collection of all upper and lower semi-Fredholm operators from $X$ into $Y$ will be denoted by $\Phi_+(X, Y)$ and $\Phi_-(X, Y)$, respectively. Obviously, the identity $\Phi(X, Y) = \Phi_-(X, Y) \cap \Phi_+(X, Y)$ is valid and it suggests an idea that, first of all, conditions of necessary and sufficient should be found under which either the inclusion $T \in \Phi_-(\ell_\infty)$ or the inclusion $T \in \Phi_+(\ell_\infty)$ holds. Below the second and third sections are devoted to these problems, respectively. In particular, the conditions under which $R(T)$ is closed should be investigated and the results in this direction are included in Section 2. This section has two subsections. In the first one, we consider the general case of an arbitrary operator $T$ on $\ell_\infty$. When we specialize the Fredholm properties of operators to the class of almost diagonal operators, we obtain much sharper results. This task is taken in Section 2.2.

By analogy to the notion of Fredholm sequence, we introduce the following two notions.

**Definition 1.5** A bounded sequence $\{x_n^*\}$ in $\ell_\infty^*$ is said to be lower (upper) semi-Fredholm if the operator $T$ defined by $\{x_n^*\}$ is lower (upper) semi-Fredholm.

A compact perturbation of a Fredholm operator and a lower (upper) semi-Fredholm operator remains Fredholm, that is, for example, the relations $T \in \Phi(X, Y)$ and $K \in \mathcal{K}(X, Y)$ imply $T + K \in \Phi(X, Y)$; in particular, for every sequence $\{z_n^*\}$ in $\ell_\infty^*$ such that the set $\{z_1^*, z_2^*, \cdots\}$ is relatively compact and for every Fredholm sequence $\{x_n^*\}$ the sequence $\{x_n^* + z_n^*\}$ is also Fredholm. In particular, a Fredholm sequence can be changed on a finite number of elements and still remains Fredholm.