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A MODIFIED TIKHONOV REGULARIZATION METHOD FOR THE CAUCHY PROBLEM OF LAPLACE EQUATION*

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Abstract In this paper, we consider the Cauchy problem for the Laplace equation, which is severely ill-posed in the sense that the solution does not depend continuously on the data. A modified Tikhonov regularization method is proposed to solve this problem. An error estimate for the a priori parameter choice between the exact solution and its regularized approximation is obtained. Moreover, an a posteriori parameter choice rule is proposed and a stable error estimate is also obtained. Numerical examples illustrate the validity and effectiveness of this method.

Key words Cauchy problem for Laplace equation; ill-posed problem; a posteriori parameter choice; error estimate

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1 Introduction

The Cauchy problem for the Laplace equation has been extensively investigated in many practical areas. For example, some problems related to the geophysics [1, 2], plasma physics [3], cardiology [4], bioelectric field problems [5] and non-destructive testing [6]. In the Hadamard's famous paper [7], this problem is firstly introduced as a classical example of ill-posed problems, which shows that any small change of the data may cause dramatically large errors in the solution. Thus, it is impossible to solve this problem using classical numerical methods and requires special techniques, e.g., regularization. There have been many papers devoted to this subject, such as the Fourier method [8], the central difference method [9], the quasi-reversibility method [10–13], the Tikhonov regularization method [14], the conjugate gradient method [15], the moment method [16–18], the wavelet method [19–20], the mollification method [21], the fundamental solutions [22], and etc.

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In this paper, we consider the following Cauchy problem for the Laplace equation in a strip domain [23]:

$$\begin{cases} \triangle u(x,y)=0, & x\in(0,1), y\in\mathbb{R},\\ u(0,y)=0, & y\in\mathbb{R},\\ u_x(0,y)=\varphi(y), & y\in\mathbb{R}, \end{cases} \tag{1.1}$$

the solution $u(x,y)$ for $0<x<1$ will be determined from the noisy data $\varphi^\delta(y)$ which satisfies

$$\|\varphi^\delta-\varphi\|_{L^2(\mathbb{R})}\le\delta. \tag{1.2}$$

Problem (1.1) is severely ill-posed. However, to our knowledge, there are few papers devoted to the error estimates of regularization methods. In this paper, we will consider not just the a priori choice of the regularization parameter for the modified Tikhononv regularization method, but also the a posteriori choice of the regularization parameter will be given for problem (1.1).

The modified Tikhonov regularization was based on the Tikhonov regularization method. Skillfully modified the penalty term of the Tikhonov functional, a better filter which filters the high frequencies of the measured data was obtained. This idea initially came from Carasso, who modified the filter gained by the Tikhonov regularization method, and the order optimal error estimate was obtained in [24]. By this method, Zhao [25] considered backward heat equation, Fu [26] considered the inverse heat conduction problem on a general sideways parabolic equation. Feng [27] used this method to consider the Cauchy problem for the Helmholtz equation. Cheng [28, 29] used this method to consider the spherically symmetric inverse problem. Yang [30, 31] used this method to consider the identification unknown source.

The plan of this paper is as follows. In Section 2 we simply analyze the ill-posedness of the Cauchy problem for Laplace equation and propose the modified Tikhonov regularization method. In Section 3 we provide some stability and convergence estimates for the Cauchy problem. In Section 4, some numerical examples are proposed to show the effectiveness of this method. Section 5 puts an end to this paper with a brief conclusion.

2 Preliminaries

In this section, we give some auxiliary results which is needed in next section. For $g(y)\in L^2(\mathbb{R})$, $\hat{g}(\xi)$ denotes its Fourier transform defined by

$$\hat{g}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-\mathrm{i}\xi y}g(y)\mathrm{d}y. \tag{2.1}$$

Let $\|\cdot\|$ denote the norm in $L^2(\mathbb{R})$. Then there holds the Parseval formula:

$$\|g(\cdot)\|=\|\hat{g}(\cdot)\|. \tag{2.2}$$

The application of the Fourier transform technique to problem (1.1) with respect to the variable y yields the following problem in frequency space:

$$\begin{cases} \hat{u}_{xx}(x,\xi)-\xi^2\hat{u}(x,\xi)=0, & x\in(0,1), \xi\in\mathbb{R},\\ \hat{u}(0,\xi)=0, & \xi\in\mathbb{R},\\ \hat{u}_x(0,\xi)=\hat{\varphi}(\xi), & \xi\in\mathbb{R}. \end{cases} \tag{2.3}$$