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A NOTE ON THE REPRESENTATIONS FOR THE GENERALIZED DRAZIN INVERSE OF BLOCK MATRICES*

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Abstract We present some representations for the generalized Drazin inverse of a block matrix $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in a Banach algebra $\mathcal{A}$ in terms of a^d and $(bc)^d$ under certain conditions, extending some recent result related to the generalized Drazin inverse of an anti-triangular operator matrix. Also, several particular cases of this result are considered.

Key words generalized Drazin inverse; block matrix; Banach algebra

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1 Introduction

Let $\mathcal{A}$ be a complex unital Banach algebra with unit 1. For $a \in \mathcal{A}$, we denote by $\sigma(a)$ the spectrum of a. The sets of all invertible, nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$, $\mathcal{A}^{\rm nil}$ and $\mathcal{A}^{\rm qnil}$, respectively.

Let us recall that the generalized Drazin inverse of $a \in \mathcal{A}$ (or Koliha–Drazin inverse of a) is the unique element $a^d \in \mathcal{A}$ which satisfies

$$a^d a a^d = a^d, \qquad a a^d = a^d a, \qquad a - a^2 a^d \in \mathcal{A}^{\rm qnil}.$$

The generalized Drazin inverse a^d exists if and only if $0 \notin \operatorname{acc} \sigma(a)$ (see [1]). It is well-known that $a^\pi = 1 - aa^d$ is the spectral idempotent of a corresponding to the set $\{0\}$. We use $\mathcal{A}^d$ to denote the set of all generalized Drazin invertible elements of $\mathcal{A}$.

If the element $a - a^2 a^d$ is nilpotent in the above definition, then a^d is ordinary Drazin inverse. The group inverse, denoted by $a^\#$, is a special case of the Drazin inverse for which $a - a^2 a^d \in \mathcal{A}^{\rm nil}$ is nilpotent is replaced with $a = aa^d a$. By $\mathcal{A}^\#$ will be denoted the set of all group invertible elements of $\mathcal{A}$. In this paper we agree that $a^0 = 1$, $0^k = 0$ and $\sum\limits_{k=1}^{n-j} \star = 0$ for $n \le j$, where $k, j, n \in \mathbf{N}$.

The following result is well-known for matrices [2, Theorem 2.1], for bounded linear operators [3, Theorem 2.3] and for elements of Banach algebra [4].

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Lemma 1.1 ([4, Example 4.5]) Let $a, b \in \mathcal{A}^d$ and let $ab = 0$. Then $a + b \in \mathcal{A}^d$ and

$$(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n (a^d)^{n+1}.$$

If $p = p^2 \in \mathcal{A}$ is an idempotent, we can represent element $a \in \mathcal{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

We present well-known result on the generalized Drazin inverse of a triangular block matrix.

Lemma 1.2 ([4, Theorem 2.3]) Let

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \in \mathcal{A}$$

relative to the idempotent $p \in \mathcal{A}$. If $a \in (p\mathcal{A}p)^d$ and $b \in ((1-p)\mathcal{A}(1-p))^d$, then $x \in \mathcal{A}^d$ and

$$x^d = \begin{bmatrix} a^d & 0 \\ u & b^d \end{bmatrix},$$

where

$$u = \sum_{n=0}^{\infty} (b^d)^{n+2} c a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n c (a^d)^{n+2} - b^d c a^d.$$

We state the auxiliary results which are proved for matrices [5] and Banach space operators [6], and they are equally true for elements of Banach algebras.

Lemma 1.3 Let $p \in \mathcal{A}$ be an idempotent, $b \in p\mathcal{A}(1-p)$ and $c \in (1-p)\mathcal{A}p$. If $bc \in (p\mathcal{A}p)^d$, then $cb \in ((1-p)\mathcal{A}(1-p))^d$, $(cb)^d = c[(bc)^d]^2 b$ and $b(cb)^d = (bc)^d b$.

Lemma 1.4 Let $x = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \in \mathcal{A}$ relative to the idempotent $p \in \mathcal{A}$. Then $x \in \mathcal{A}^d$ if and only if $bc \in (p\mathcal{A}p)^d$ (or $cb \in ((1-p)\mathcal{A}(1-p))^d$). In this case,

$$x^d = \begin{bmatrix} 0 & (bc)^d b \\ c(bc)^d & 0 \end{bmatrix} = \begin{bmatrix} 0 & b(cb)^d \\ (cb)^d c & 0 \end{bmatrix}.$$

Properties of the Drazin inverse of an operator or a matrix and its applications to singular differential equations and singular difference equations, iterative methods, Markov chains can be found in [7]. Campbell and Meyer [7] proposed the problem of finding an explicit representation for the Drazin inverse of a 2×2 block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in terms of its sub-blocks, where A and D are required to be square matrices. Many authors have considered this problem, under different conditions on the individual blocks [2, 8–13].

In [6], several formulaes for the generalized Drazin inverse of an anti-triangular operator matrix $M = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ are given under some conditions on Banach spaces operators A, B, C.