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Structured sublinear compressive sensing via belief propagation

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ABSTRACT

Compressive sensing (CS) is a sampling technique designed for reducing the complexity of sparse data acquisition. One of the major obstacles for practical deployment of CS techniques is the signal reconstruction time and the high storage cost of random sensing matrices. We propose a new structured compressive sensing scheme, based on codes of graphs, that allows for a joint design of structured sensing matrices and logarithmic-complexity reconstruction algorithms. The compressive sensing matrices can be shown to offer asymptotically optimal performance when used in combination with orthogonal matching pursuit (OMP) methods. For reduced-complexity greedy reconstruction schemes, we propose a new family of list-decoding belief propagation algorithms, as well as reinforced and multiple-basis belief propagation (BP) algorithms. Our simulation results indicate that reinforced BP CS schemes offer very good complexity–performance tradeoffs for very sparse signal vectors.

1. Introduction

Compressive sensing (CS) has received significant attention due to various applications in signal processing, networking, MRI data acquisition, bioinformatics, and remote sensing [1]. CS is a sampling technique for compressible and/or K-sparse signals, i.e., signals that can be represented by $K \ll N$ significant coefficients over an N-dimensional basis. Sampling of a K-sparse, discrete-time signal $\mathbf{x}$ of dimension N is accomplished by computing a measurement vector, $\mathbf{y}$, that consists of $m \ll N$ linear projections, i.e.,

$$\mathbf{y} = \mathbf{\Phi}\mathbf{x}.$$

Here, $\mathbf{\Phi}$ represents an $m \times N$ matrix, usually over the field of real numbers [2]. Although the reconstruction of the signal $\mathbf{x} \in \mathbb{R}^N$ from the possibly noisy random projections is an ill-posed task, the prior knowledge of signal sparsity allows for recovering $\mathbf{x}$ in polynomial time using $m \ll N$ observations only. If the reconstruction problem is cast as an ℓ_0 minimization problem [3], it can be shown that, in order to reconstruct a K-sparse signal $\mathbf{x}$, ℓ_0 minimization requires only $m = 2K$ random projections. In this setting, it is assumed that the signal and the measurements are noise free. Unfortunately, the ℓ_0 optimization problem is a combinatorial problem that for general instances of sensing is NP-hard. The work by Donoho and Candes et al. [1,2,4,5] demonstrated that CS reconstruction is a polynomial time problem—conditioned on the constraint that more than $2K$ measurements are used. The key idea behind their approach is that it is not necessary to resort to ℓ_0 optimization to recover $\mathbf{x}$ from the under-determined inverse problem: a tractable ℓ_1 optimization, based on linear programming (LP) techniques, yields an equivalent solution provided that the sensing matrix $\mathbf{\Phi}$ satisfies the so-called *restricted isometry property* (RIP), with a constant RIP parameter.

While LP techniques play an important role in designing computationally efficient CS decoders, their complexity renders them highly impractical for many applications. In such cases, the need for fast reconstruction algorithms – preferably operating in time linear in N, and without significant performance loss compared to LP methods – is of critical importance. A common approach to mitigating these problems is to increase the number of measurements and to use greedy reconstruction methods.

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Several classes of low-complexity reconstruction techniques have been put forward as alternatives to linear programming (LP) recovery, including group testing methods [6], pursuit strategies such as orthogonal matching pursuit (OMP), subspace pursuit (SP) and compressive sampling matching pursuit (CoSaMP) [7–10], and coding-theoretic techniques [11–13].

We focus our attention on two intertwined problems related to low-complexity CS reconstruction techniques. The first problem is concerned with designing structured matrices that provide RIP-type performance guarantees, since such matrices have low storage complexity and may potentially yield to faster reconstruction approaches. The second problem is concerned with how to most efficiently exploit the structure of the sensing matrix in order to further reduce the reconstruction complexity of greedy-like methods that use correlation maximization as one of their key steps. The solution we propose addresses both these issues, and can be succinctly described as follows.

It is known that random Bernoulli matrices – matrices with i.i.d. Bernoulli$(1/2)$ distributed entries – have constant RIP parameters with a number of measurements proportional to $K\log(N/K)$ [1,5]. This number of measurements suffices for exact reconstruction of K-sparse signals using LP methods. One property of Bernoulli matrices is that, for sufficiently large dimensions, the fraction of the symbols $+1$ and -1 per row and per column is close to one half. Furthermore, a similar property holds for any sufficiently large submatrix of the matrix. Consequently, one approach for designing structured compressive sensing may be reduced to mimicking this property of Bernoulli matrices and then showing that the matrices indeed have a constant RIP parameter.

This task can be accomplished via linear error-correcting coding. Due to the linear structure of the code, using codewords of a binary linear code with zeros replaced by $+1$s and ones by -1s as columns of the matrix ensures the row-weight balancing property. Furthermore, if the weight of the codewords is chosen close to half of the codelength, similar concentration results will hold for the columns of the sensing matrix.

The idea of using linear error-correcting codes was first proposed in [14], where encodings of Reed–Muller codewords were used for columns of a compressive sensing matrix [15,16]. The authors proposed independently a similar framework based on low-density parity-check codes [17] in [18], and some follow-up results on this work were reported in [19]. Another approach for constructing sensing matrices by trying to match their distribution of singular values to that of Bernoulli matrices was put forward in [20,21].

The advantage of using sensing matrices based on error-control codes from the perspective of reconstruction complexity is best explained in the context of greedy algorithms, as argued in our earlier work [18]. A key step of greedy reconstruction algorithms is to compute the correlations of the observed vector $\mathbf{y}$ with the columns of the sensing matrix $\mathbf{\Phi}$ and to identify the column with the largest correlation. When the columns of the matrix represent codewords of a linear code, this problem reduces to the extensively studied maximum likelihood (ML) decoding problem. For certain classes of codes, near-ML decoding can be performed in time linear in the length of the code, which in the described setting implies that near-optimal correlation optimization can be performed in time proportional to the number of rows, and not the number of columns of the sensing matrix.

We focus on codes that lead to reconstruction techniques with sublinear (more precisely, logarithmic) complexity in N. The basic construction and decoding methods are based on ideas from codes on graphs and iterative decoding. We show that a simple combination of reinforced belief propagation (BP) [22] and a novel list-decoding method can be coupled with the greedy SP algorithm to produce good reconstruction algorithms with logarithmic complexity, for the case of "super-sparse" signals previously studied in [23]. As already mentioned, the BP algorithm operates on the columns of the matrix $\mathbf{\Phi}$ of length m, and consequently its reconstruction complexity is $O(m)$.

Before outlying the organization of the paper, we would like to describe the context of our work within the vast literature on compressive sensing. Sublinear reconstruction techniques were first investigated in [23–26], while *sparse sensing matrices* coupled with BP decoding were considered in [11,25]. An idea for sublinear compressive sensing reconstruction inspired by Sudoku was described in [24], but the algorithm works only for input signals with special structural properties where one requires that all sums of subsets of coefficients are distinguishable (which is rather restrictive for binary vectors), and where the measurement matrix is random. Furthermore, the reconstruction is only partial, in that the reconstruction complexity strongly depends on the number of recovered entries of the sensed signal.

Our approach differs from all the aforementioned results in that it *does not use sparse sensing matrices* that are known to incur a performance loss compared to dense matrices, such as Bernoulli matrices. Although our structured sensing matrices are dense, they are constructed using codewords of large minimum distance low-density parity-check (LDPC) codes which themselves have sparse matrix descriptions (i.e., sparse parity-check matrices). Furthermore, no high-complexity preprocessing is required, and unlike the approach in [23], the complexity of the algorithm is not polylogarithmic in N, but only logarithmic in N; and, as opposed to using sparse matrices without RIP guarantees, our approach utilizes *structured dense matrices* constructed from sparse matrices, for which one can show asymptotic optimality with respect to the achievable coherence parameter.

The problems addressed in this paper are equally relevant to questions arising in storage and wireless communication systems, since a major part of the analysis is focused on BP decoding for channels with severe user interference. The framework proposed in this paper also allows for handling measurement noise, but the underlying results will be described elsewhere.

The paper is organized as follows. Section 2 provides a brief introduction to compressive sensing. Section 3 includes the description of a structured design approach for compressive sensing matrices $\mathbf{\Phi}$, amenable for $O(K^u\log N)$ complexity decoding of super-sparse vectors, with $u=2$